Numerical Integration of Partial Differential Equations (PDEs)

- Introduction to PDEs.
- Semi-analytic methods to solve PDEs.
- Introduction to Finite Differences.
- Stationary Problems, Elliptic PDEs.
- Time dependent Problems.
- Complex Problems in Solar System Research.

Complex Problems in Solar System Research.

- Stationary Problems:
  Magneto-hydrostatic equilibria to model magnetic field and plasma in the solar corona.
- Time-dependent Problems:
  Multi-fluid-Maxwell simulation of plasmas (courtesy Nina Elkina)

Modeling the solar corona

- Magnetic fields structure the solar corona.
- But we cannot measure them directly.
- Solution: Solve PDEs and use photospheric magnetic field measurements to prescribe boundary conditions.
- Let’s start with the simplest approach: Potential fields: $\nabla \times \mathbf{B} = 0, \; \nabla \cdot \mathbf{B} = 0$

With $\mathbf{B} = \nabla f$ we have to solve a Laplace equation:

$$\Delta f = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial\phi^2}=0$$

We try to solve this equation by separation of variables:

$$f(r,\theta,\phi)=f_1(r)\cdot f_2(\theta,\phi)$$

and after multiplication with $\frac{r^2}{f_1(r)f_2(\theta,\phi)}$ we get:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial f_1(r)}{\partial r}\right) = l(l+1)f_1(r)$$

$$\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f_2}{\partial\theta}\right)+\frac{1}{\sin^2(\theta)}\frac{\partial^2 f_2}{\partial\phi^2} = -l(l+1)f_2(\theta,\phi)$$

The solutions of the radial part are:

$$f_1(r)=\boxed{r^{-(l+1)},}\ \text{and}\ f_1(r)=\boxed{r^l}$$

We can further separate the angular equation (and get another separation constant m) or just look in a text-book or Wikipedia and find that this equation is solved by spherical harmonics $\boxed{Y_{lm}(\theta,\phi)}$

The 3D-solution of the Laplace equation can be found by superposition of the particular solutions $f(r,\theta,\phi) = f_1(r) \cdot f_2(\theta,\phi)$ as:

$$\boxed{f(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm}\, r^l + B_{lm}\, r^{-(l+1)}\right]\, Y_{lm}(\theta,\phi)}$$

where Y_{lm} are Spherical Harmonics and A_{lm} and B_{lm} are coefficient which we prescribe from boundary conditions.

In the photosphere $(r = 1R_s)$ the radial magnetic field $B_r(r = 0)$ is measured and used to prescribe von Neumann B.C. We make a spherical harmonic decomposition:

$$B_r(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} Y_{lm}(\theta,\phi)$$

$$C_{lm} = \int_0^{2\pi} \int_0^{\pi} Y^*_{lm}(\theta,\phi)\; B_r(\theta,\phi)\; \sin(\theta)\, d\theta d\phi$$

where $Y^*_{lm} = (-1)^m\, Y_{l,-m}$.

Outer radial boundary at source surface ($r_1 \approx 2.5 R_s$). We assume that the field becomes radial here: $\vec{B} = B_r \, \vec{e}_r$ for $r = r_1$:

$$B_\theta = \frac{1}{r}\frac{\partial f(r,\theta,\phi)}{\partial\theta}$$
$$B_\phi = \frac{1}{r\,\sin(\theta)}\frac{\partial f(r,\theta,\phi)}{\partial\phi}$$

are supposed to vanish at $r = r_1$.
Together with the photospheric boundary condition we get two equation to calculate A_{lm} and B_{lm}:

$$A_{lm}\, l\, r_0^{(l-1)} - B_{lm}\,(l+1)\, r_0^{-(l+2)} = C_{lm}$$
$$A_{lm}\, r_1^l + B_{lm}\, r_1^{-(l+1)} = 0$$

which lead to:

$$A_{lm} = \frac{C_{lm}\, {r_0}^{2+l}}{{r_1}^{1+2\,l} + l\ \left({r_0}^{1+2\,l} + {r_1}^{1+2\,l}\right)}$$
$$B_{lm} = -\left(\frac{C_{lm}\, {r_0}^{2+l}\, {r_1}^{1+2\,l}}{{r_1}^{1+2\,l} + l\ \left({r_0}^{1+2\,l} + {r_1}^{1+2\,l}\right)}\right)$$

Solution of Laplace equation for potential coronal magnetic fields:

$$f(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm}\, r^{l} + B_{lm}\, r^{-(l+1)}\right] \, Y_{lm}(\theta,\phi)$$

$$B_r = \frac{\partial f}{\partial r}$$

$$B_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$

$$B_\phi = \frac{1}{r\,\sin(\theta)}\frac{\partial f}{\partial \phi}$$

Show example in IDL

Nonlinear Force-Free Fields

- Potential fields give impression about global topology of the coronal magnetic field.
- But: Approach is to simple to describe magnetic field and energy in active regions accurately.
- We include field aligned electric currents, the (nonlinear) force-free approach.

$$\begin{aligned}(\nabla \times \mathbf{B}) \times \mathbf{B} &= \mathbf{0} \\ \nabla \cdot \mathbf{B} &= 0\end{aligned} \quad \overset{\text{Equivalent}}{\Longleftrightarrow} \quad \begin{aligned}\nabla \times \mathbf{B} &= \alpha \mathbf{B} \\ \mathbf{B} \cdot \nabla \alpha &= 0\end{aligned}$$

Nonlinear Force-Free Fields
(direct upward integration)

$$\boxed{\begin{aligned}\nabla \times \mathbf{B} &= \alpha \mathbf{B} \\ \mathbf{B} \cdot \nabla \alpha &= 0\end{aligned}}$$

Wu et al. 1990 proposed to solve these equations by upward integration:

- Compute vertical current in photosphere (z=0)

$$\mu_0 j_{z0} = \frac{\partial B_{y0}}{\partial x} - \frac{\partial B_{x0}}{\partial y}$$

- Compute alpha

$$\alpha_0 = \frac{j_{z0}}{B_{z0}}$$

- compute horizontal currents

$$j_{x0} = \alpha_0 \, B_{x0}, \quad j_{y0} = \alpha_0 \, B_{y0}$$

- Integrate B upwards
- Repeat all steps for z=1,2,...

$$\frac{\partial B_{x0}}{\partial z} = j_{y0} + \frac{\partial B_{z0}}{\partial x},$$

$$\frac{\partial B_{y0}}{\partial z} = \frac{\partial B_{z0}}{\partial y} - j_{x0},$$

$$\frac{\partial B_{z0}}{\partial z} = -\frac{\partial B_{x0}}{\partial x} - \frac{\partial B_{y0}}{\partial y}.$$

Nonlinear Force-Free Fields
(direct upward integration)

- Straight forward scheme.
- Easy to implement.
- But: Not useful because the method is unstable.
- Why?
- Ill-posed problem.

$$
\begin{aligned}
\frac{\partial B_{x0}}{\partial z} &= j_{y0} + \frac{\partial B_{z0}}{\partial x}, \\
\frac{\partial B_{y0}}{\partial z} &= \frac{\partial B_{z0}}{\partial y} - j_{x0}, \\
\frac{\partial B_{z0}}{\partial z} &= -\frac{\partial B_{x0}}{\partial x} - \frac{\partial B_{y0}}{\partial y}.
\end{aligned}
$$

Why is the problem ill-posed?

- Problem-1: Measured Magnetic field in photosphere is not force-free consistent.
- Cure: We do regularization (or preprocessing) to prescribe consistent boundary conditions.
- Problem-2: Even for 'ideal consistent' data the upward integration is unstable (exponential growing modes blow up solution).
- Cure: Reformulate the equations and apply a stable (iterative) method.

Consistency criteria for boundary-data (Aly 1989)

If these relations are NOT fulfilled, then the boundary data are **inconsistent** with the nonlinear force-free PDEs.

Ill posed Problem.

Preprocessing or Regularization
(Wiegelmann et al. 2006)

Input: Measured ill posed data => **Output**: Consistent B.C.

$$L_{cp} = \mu_1 L_1 + \mu_2 L_2 + \mu_3 L_3 + \mu_4 L_4$$

$$L_1 = \left(\sum_p B_x B_z \right)^2 + \left(\sum_p B_y B_z \right)^2 + \left(\sum_p B_z^2 - B_x^2 - B_y^2 \right)^2$$

$$L_2 = \left(\sum_p x (B_z^2 - B_x^2 - B_y^2) \right)^2 + \left(\sum_p y (B_z^2 - B_x^2 - B_y^2) \right)^2 + \left(\sum_p (y B_x B_z - x B_y B_z) \right)^2$$

$$L_3 = \sum_p (B_x - B_{xobs})^2 + \sum_p (B_y - B_{yobs})^2 + \sum_p (B_z - B_{zobs})^2$$

$$L_4 = \sum_p (\Delta B_x)^2 + \sum_p (\Delta B_y)^2 + \sum_p (\Delta B_z)^2$$

Non-linear Force-Free Fields

Force-free magnetic fields have to obey

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{0}, \ \nabla \cdot \mathbf{B} = 0$$

We define the functional (Wheatland, Sturrock, Roumeliotis 2000)

$$L = \int_V w(x, y, z) \ \left[B^{-2} \left|(\nabla \times \mathbf{B}) \times \mathbf{B}\right|^2 + \left|\nabla \cdot \mathbf{B}\right|^2\right] \ d^3x$$

w is a weighting function (Wiegelmann 2004).
We minimize L:

$$\frac{1}{2} \frac{dL}{dt} = - \int_V \frac{\partial \mathbf{B}}{\partial t} \cdot \tilde{\mathbf{F}} \ d^3x - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \tilde{\mathbf{G}} \ d^2x$$

If all components of $\mathbf{B}$ are fixed on the boundaries of a computational box we get an evolution equation for $\mathbf{B}$

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \mu \tilde{\mathbf{F}}}$$

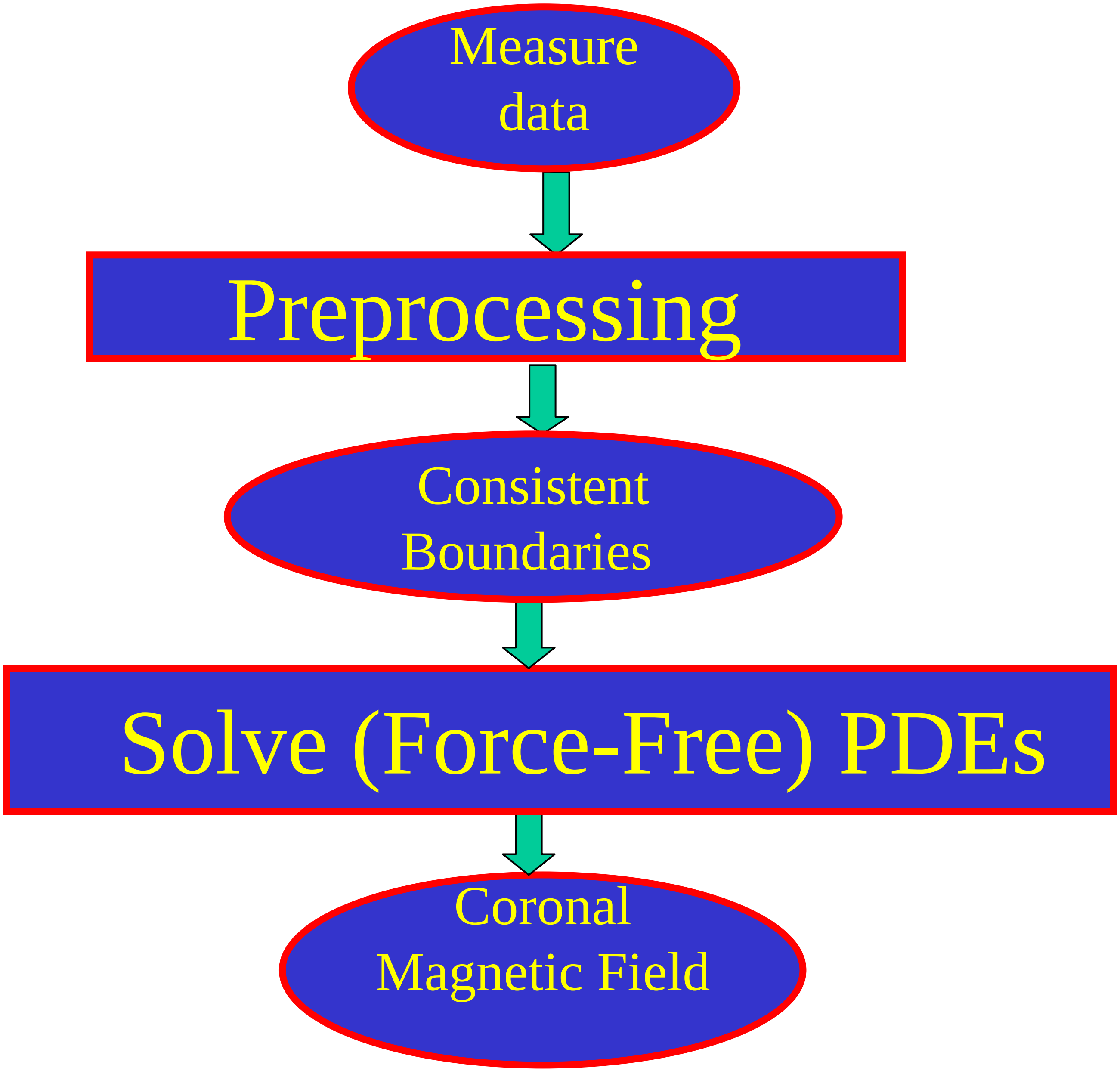
$$\begin{aligned}\tilde{\mathbf{F}} &= w\,\mathbf{F} + (\boldsymbol{\Omega}_{\mathrm{a}} \times \mathbf{B}) \times \nabla w + (\boldsymbol{\Omega}_{\mathrm{b}} \cdot \mathbf{B})\,\nabla w \\ \tilde{\mathbf{G}} &= w\,\mathbf{G}\end{aligned}$$

$$\begin{aligned}\mathbf{F} &= \nabla \times (\boldsymbol{\Omega}_{\mathrm{a}} \times \mathbf{B}) - \boldsymbol{\Omega}_{\mathrm{a}} \times (\nabla \times \mathbf{B}) \\ &\quad + \nabla(\boldsymbol{\Omega}_{\mathrm{b}} \cdot \mathbf{B}) - \boldsymbol{\Omega}_{\mathrm{b}}(\nabla \cdot \mathbf{B}) + (\boldsymbol{\Omega}_{\mathrm{a}}^2 + \boldsymbol{\Omega}_{\mathrm{b}}^2)\,\mathbf{B}\end{aligned}$$

$$\mathbf{G} = \hat{\mathbf{n}} \times (\boldsymbol{\Omega}_{\mathrm{a}} \times \mathbf{B}) - \hat{\mathbf{n}}(\boldsymbol{\Omega}_{\mathrm{b}} \cdot \mathbf{B}),$$

$$\boldsymbol{\Omega}_{\mathrm{a}} = B^{-2}\,\left[(\nabla \times \mathbf{B}) \times \mathbf{B}\right]$$

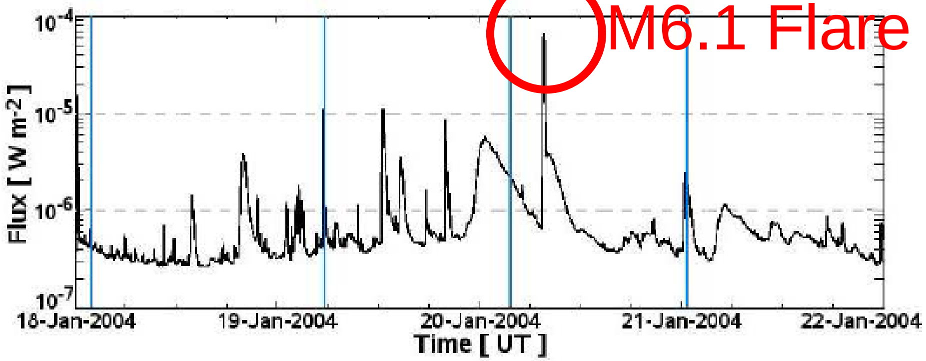
$$\boldsymbol{\Omega}_{\mathrm{b}} = B^{-2}\,\left[(\nabla \cdot \mathbf{B})\,\mathbf{B}\right].$$

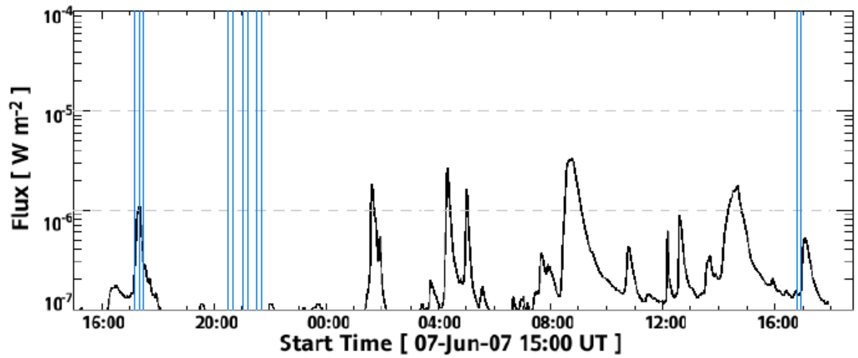

Measure data
Preprocessing
Consistent Boundaries
Solve (Force-Free) PDEs
Coronal Magnetic Field

Flaring Active Region

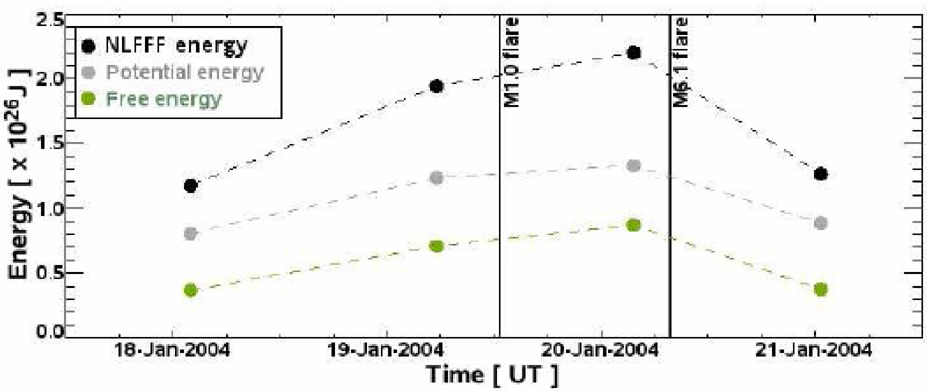
(Thalmann & Wiegelmann 2008)

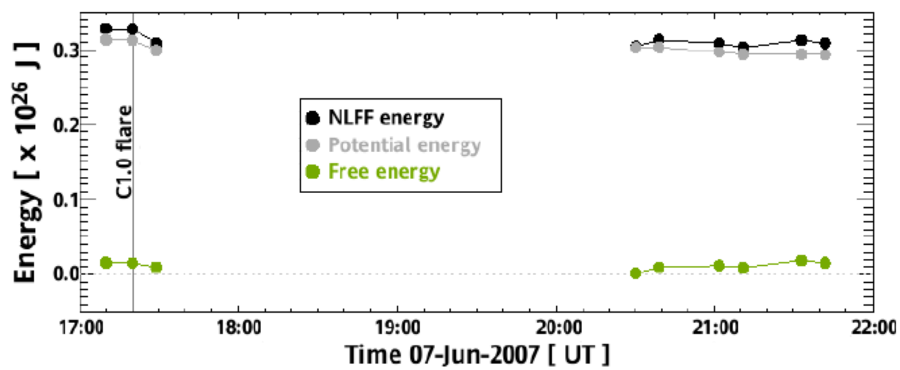
Quiet Active Region





Solar X-ray flux. Vertical blue lines: vector magnetograms available





Magnetic field extrapolations from Solar Flare telescope

Extrapolated from SOLIS vector magnetograph

Magnetohydrostatics

Model magnetic field and plasma consistently:

$$\underbrace{(\nabla \times \mathbf{B}) \times \mathbf{B}}_{\textbf{Lorentz force}} - \underbrace{\mu_0 \nabla p}_{\textbf{pressure gradient}} - \underbrace{\mu_0 \rho \nabla \Psi}_{\textbf{gravity}} = \mathbf{0}$$

$$\nabla \cdot \mathbf{B} = 0$$

We define the functional

$$L(\mathbf{B}, p, \rho) = \int \left[\frac{|(\nabla \times \mathbf{B}) \times \mathbf{B} - \mu_0 \nabla p - \mu_0 \rho \nabla \Psi|^2}{B^2} + |\nabla \cdot \mathbf{B}|^2 \right] r^2 \sin\theta \, dr \, d\theta \, d\phi$$

The magnetohydrostatic equations are fulfilled if L=0

For easier mathematical handling we use

$$\mathbf{\Omega_a} = B^{-2} \left[(\nabla \times \mathbf{B}) \times \mathbf{B} - \mu_0 \nabla p - \mu_0 \rho \nabla \Psi \right]$$
$$\mathbf{\Omega_b} = B^{-2} \left[(\nabla \cdot \mathbf{B}) \, \mathbf{B} \right],$$

and rewrite L as

$$L = \int_V B^2 \Omega_a^2 + B^2 \Omega_b^2 \, r^2 \sin\theta \, dr \, d\theta \, d\phi.$$

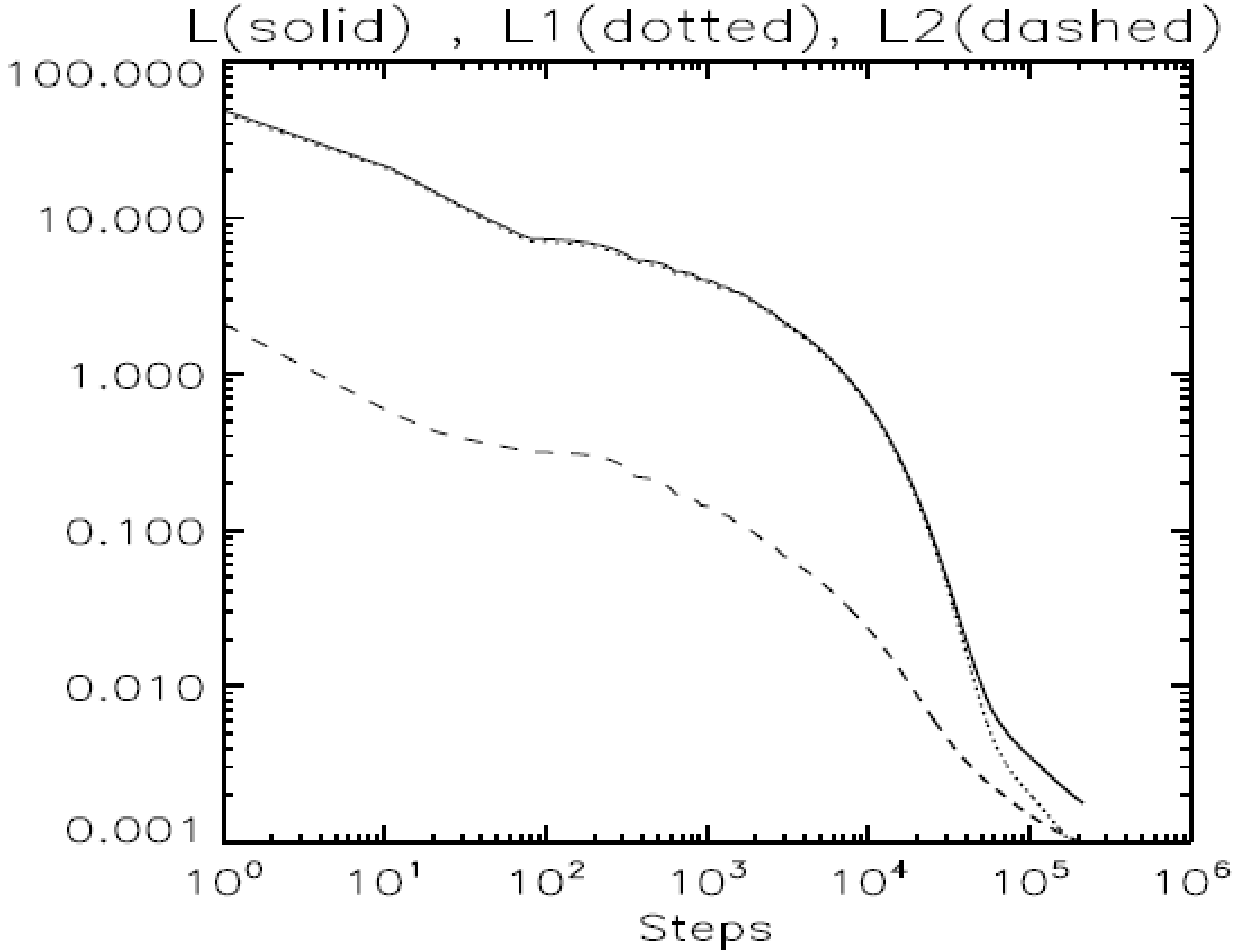
Taking the derivative of L with respect to an iteration parameter t, where $\mathbf{B}$, p, ρ are assumed to depend upon t, we obtain

$$\begin{aligned}
\frac{1}{2}\,\frac{dL}{dt} = & -\int_V \frac{\partial \mathbf{B}}{\partial t}\cdot \mathbf{F}\, dV + \int_V \frac{\partial p}{\partial t}\,\mu_0 \nabla\cdot \mathbf{\Omega_a}\, dV \\
& -\int_V \frac{\partial \rho}{\partial t}\,\mu_0 \mathbf{\Omega_a}\cdot \nabla\Psi\, dV - \cancel{\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot \mathbf{G}\, dS} \\
& - \cancel{\int_S \frac{\partial p}{\partial t}\,\mu_0 \mathbf{\Omega_a}\cdot d\mathbf{S}},
\end{aligned}$$

$$\left.\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= \mu\, \mathbf{F} \\
\frac{\partial p}{\partial t} &= -\nu\, \mu_0 \nabla\cdot \mathbf{\Omega_a} \\
\frac{\partial \rho}{\partial t} &= \xi\, \mu_0\, \mathbf{\Omega_a}\cdot \nabla\Psi
\end{aligned}\right\}$$

Iterative Equations ensure monotonously decreasing functional L for vanishing surface integrals (boundary conditions).

$$L(\mathbf{B}, p, \rho) = \int \left[\frac{|(\nabla \times \mathbf{B}) \times \mathbf{B} - \mu_0 \nabla p - \mu_0 \rho \nabla \Psi|^2}{B^2} + |\nabla \cdot \mathbf{B}|^2 \right] r^2 \sin\theta \, dr \, d\theta \, d\phi$$



Modeling the solar corona Summary

- First one has to **find appropriate PDEs** which are adequate to model (certain aspects of) the solar corona. Here: Stationary magnetic fields and plasma.
- Use measurements to prescribe boundary conditions.
- **Regularize** (preprocess) **data** to derive **consistent boundary conditions** for the chosen PDE.
- **Stationary equilibria** (solution of our PDEs) can be used as initial condition for time dependent computation of other PDEs (MHD-simulations, planned).

Multi-fluid-Maxwell simulation of plasmas (courtesy Nina Elkina)

- The kinetic Vlasov-Maxwell system.
- From 6D-Vlasov equation to 3D-fluid approach.
- Generalization of flux-conservative form.
- Lax-Wendroff + Slope limiter
- Application: Weibel instability

Kinetic approach for collisionless plasma

Vlasov equation for plasma species

$$\frac{df}{dt} = \frac{\partial f_\alpha}{\partial t} + \vec{v}\frac{\partial f_\alpha}{\partial \vec{r}} + \frac{q_\alpha}{m_\alpha}\left[\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right]\frac{\partial f_\alpha}{\partial \vec{v}} = 0$$

Maxwell equations for EM fields

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B} - \frac{4\pi}{c}\sum_\alpha q_\alpha \int \vec{v} f_\alpha d\vec{v} \qquad \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

$$\nabla \cdot \vec{E} = 4\pi \sum_\alpha q_\alpha \int f_\alpha d\vec{v} \qquad \nabla \cdot \vec{B} = 0$$

$$f(x, y, z, v_x, v_y, v_z, t)$$

3D + 3V = 6 dimensions+time

How to loose information?

Instead of all the details of the distribution of particles consider only a small number of velocity moments:

Density:

$$n(x,t) = \int dv\; F = \sum_{i=1,N} \delta(x - x_i)$$

Momentum density:

$$n(x,t)u(x,t) = \int dv\; v\; F = \sum_{i=1,N} v_i\; \delta(x - x_i)$$

Kinetic energy density:

$$K(x,t) = \int dv\; \frac{m}{2} v^2\; F = \sum_{i=1,N} \frac{m}{2} v_i^2\; \delta(x - x_i)$$

Kinetic energy flux

$$Q(x,t) = \int dv\; \frac{m}{2} v^3\; F = \sum_{i=1,N} \frac{m}{2} v_i^3\; \delta(x - x_i)$$

etc…

The multifluid simulation code

$$
\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ \rho v_x v_x + P_{xx} \\ \rho v_x v_y + P_{xy} \\ \rho v_x v_z + P_{xz} \\ \rho v_y v_y + P_{yy} \\ \rho v_y v_z + P_{yz} \\ \rho v_z v_z + P_{zz} \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho v_x \\ \rho v_x v_x + P_{xx} \\ \rho v_x v_y + P_{xy} \\ \rho v_x v_z + P_{xz} \\ \rho v_x v_x v_x + 3 v_x P_{xx} \\ \rho v_x v_x v_y + 2 v_x P_{xy} + v_y P_{xx} \\ \rho v_x v_x v_z + 2 v_x P_{xz} + v_z P_{xx} \\ \rho v_x v_y v_y + v_x P_{yy} + 2 v_y P_{xy} \\ \rho v_x v_y v_z + v_x P_{yz} + v_y P_{xz} + v_z P_{xy} \\ \rho v_x v_z v_z + v_x P_{zz} + 2 v_z P_{xz} \end{pmatrix} = \frac{q}{m}\begin{pmatrix} 0 \\ \rho(E_x + v_y B_x - v_z B_y) \\ \rho(E_y + v_z B_x - v_x B_z) \\ \rho(E_z + v_x B_y - v_y B_x) \\ 2\rho v_x E_x + 2(B_z P_{xy} - B_y P_{xz}) \\ \rho(v_x E_y + v_y E_x) + (B_z P_{yy} - B_y P_{yz} + B_z P_{xx} + B_x P_{xz}) \\ \rho(v_x E_y + v_y E_x) + (B_z P_{yz} + B_y P_{xx} - B_y P_{zz} - B_x P_{xy}) \\ 2\rho v_x E_y + 2(B_x P_{yz} - B_z P_{xy}) \\ \rho(v_y E_z + v_z E_y) + (B_y P_{xy} - B_z P_{xz} + B_x P_{zz} - B_x P_{yy}) \\ 2\rho v_z E_z + 2(B_y P_{xz} - B_x P_{yz}) \end{pmatrix}
$$

...are solved with using high-resolution semi-discrete method. These equations include also finite Larmor radii effect, pressure anisotropy, electron inertia, charge separation

Formally the multi-fluid-equations can be written in vector form

$$\frac{\partial U}{\partial t} + \underbrace{\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}}_{\text{Fluxes}} = \underbrace{S}_{\text{Source-Term}}$$

Generalized form of our flux-conservative equation:

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\partial \mathbf{F}(\mathbf{u})}{\partial x}$$

$$\boxed{\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S}$$

The individual terms are somewhat more complex as in our example advection equation.

$$U = \begin{pmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ \rho v_x v_x + P_{xx} \\ \rho v_x v_y + P_{xy} \\ \rho v_x v_z + P_{xz} \\ \rho v_y v_y + P_{yy} \\ \rho v_y v_z + P_{yz} \\ \rho v_z v_z + P_{zz} \end{pmatrix} \quad F = \begin{pmatrix} \rho v_x \\ \rho v_x^2 + P_{xx} \\ \rho v_x v_y + P_{xy} \\ \rho v_x v_z + P_{xz} \\ 3 v_x P_{xx} + \rho v_x v_x v_x \\ (v_x P_{xy} + v_x P_{xy} + v_y P_{xx}) + \rho v_x v_x v_y \\ (v_x P_{xz} + v_x P_{xz} + v_z P_{xx}) + \rho v_x v_x v_z \\ (v_x P_{yy} + v_y P_{xy} + v_y P_{xy}) + \rho v_x v_y v_y \\ (v_x P_{yz} + v_y P_{xz} + v_z P_{xy}) + \rho v_x v_y v_z \\ (v_x P_{zz} + v_z P_{xz} + v_z P_{xz}) + \rho v_x v_z v_z \end{pmatrix}$$

$$G = \begin{pmatrix} \rho v_y \\ \rho v_y v_x + P_{xy} \\ \rho v_y v_y + P_{yy} \\ \rho v_y v_z + P_{yz} \\ (v_x P_{xy} + v_y P_{xx} + v_x P_{xy}) + \rho v_x v_x v_y \\ (v_x P_{yy} + v_y P_{xy} + v_y P_{xy}) + \rho v_x v_y v_y \\ (v_x P_{yz} + v_y P_{xz} + v_z P_{xy}) + \rho v_x v_z v_y \\ 3 v_y P_{yy} + \rho v_y v_y v_y \\ (v_y P_{yz} + v_y P_{yz} + v_z P_{yy}) + \rho v_y v_z v_y \\ (v_z P_{yz} + v_y P_{zz} + v_z P_{yz}) + \rho v_z v_z v_y \end{pmatrix} \quad S = \begin{pmatrix} 0 \\ n(E_x + v_y B_x - v_z B_y) \\ n(E_y + v_z B_x - v_x B_z) \\ n(E_z + v_x B_y - v_y B_x) \\ 2 n v_x E_x + 2(q/m)(B_z P_{xy} + B_y P_{xz}) \\ n(v_x B_y + v_y E_x) + (q/m)(B_z P_{yy} - B_y P_{yz} - B_z P_{xx} + B_x P_{xz}) \\ n(v_x E_y + v_y E_x) + (q/m)(B_z P_{yz} + B_y P_{xx} - B_y P_{zz} - B_x P_{xy}) \\ 2 n v_y E_y + 2(q/m)(B_x P_{yz} - B_z P_{xy}) \\ n(v_y E_z + v_z E_y) - (B_y P_{xy} - B_z P_{xz} + B_x P_{zz} - B_x P_{yy}) \\ 2 n v_z E_z + 2(q/m)(B_y P_{xz} - B_x P_{yz}) \end{pmatrix}$$

Multi-Fluid equations are solved together with Maxwell equations

which are written as wave-equations
(remember the first lecture, here in CGS-system):

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -4\pi\rho \qquad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\frac{4\pi}{c}\vec{J}$$

Formally we combine these equations to:

$$\frac{\partial^2 P}{\partial t^2} - \nabla^2 \vec{P} = \vec{S} \quad \text{where } P = (\phi,\, \vec{A}),\text{ and } S = (\rho,\, \vec{J})$$

Equations are solved as a system of first order equations:

$$\frac{\partial \vec{P}}{\partial t} = \vec{U} \quad \frac{\partial \vec{U}}{\partial t} = \vec{R} \quad \text{where } \vec{R} = \nabla^2 \vec{P} + \vec{S}.$$

We have to solve consistently

$$\underbrace{\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S}_{\text{Multi-Fluid equations}}$$

$$\underbrace{\frac{\partial \vec{P}}{\partial t} = \vec{U} \quad \frac{\partial \vec{U}}{\partial t} = \vec{R}}_{\text{Maxwell equations}}$$

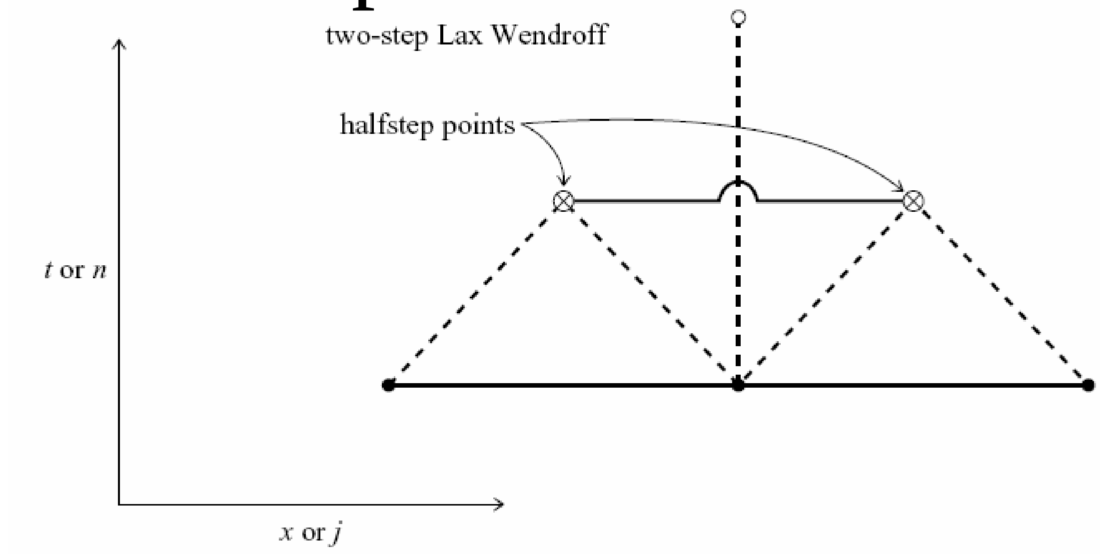
Laplace is discretized with 4th order 25-point stencil
(In earlier examples we used a 2th order 9-point stencil)

$$\nabla^2 P = \frac{1}{\Delta x^2} \begin{pmatrix} c_1 & c_2 & c_3 & c_2 & c_1 \\ c_2 & c_4 & c_5 & c_4 & c_2 \\ c_3 & c_5 & c_6 & c_5 & c_3 \\ c_2 & c_4 & c_5 & c_4 & c_2 \\ c_1 & c_2 & c_3 & c_2 & c_1 \end{pmatrix} + O(h^4)$$

$$c_1 = 0,\; c_2 = -\frac{1}{30}\, c_3 = -\frac{1}{60}\, c_4 = \frac{4}{15}\, c_5 = \frac{13}{15}\, c_6 = -\frac{21}{5}$$

Numerical scheme: Lax-Wendroff + slope limiter

- Method based on Lax-Wendroff scheme
- Additional feature: Non-oscillatory reconstruction near gradients.



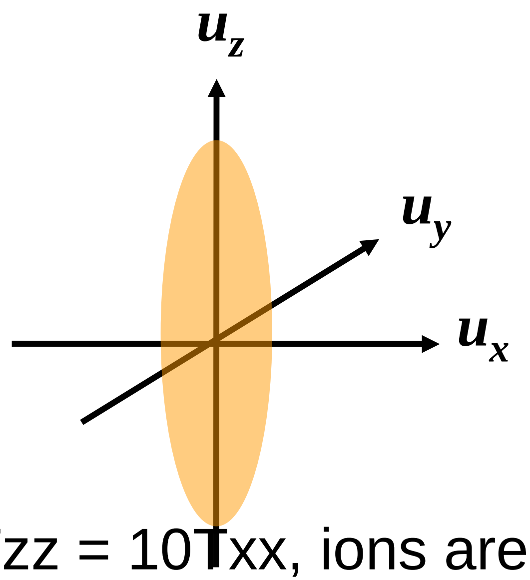
Predictor step: $$w_i^{n+1/2} = \overline{w}_i^n - \frac{\lambda}{2} F^x(w_i^n)$$

Corrector step:

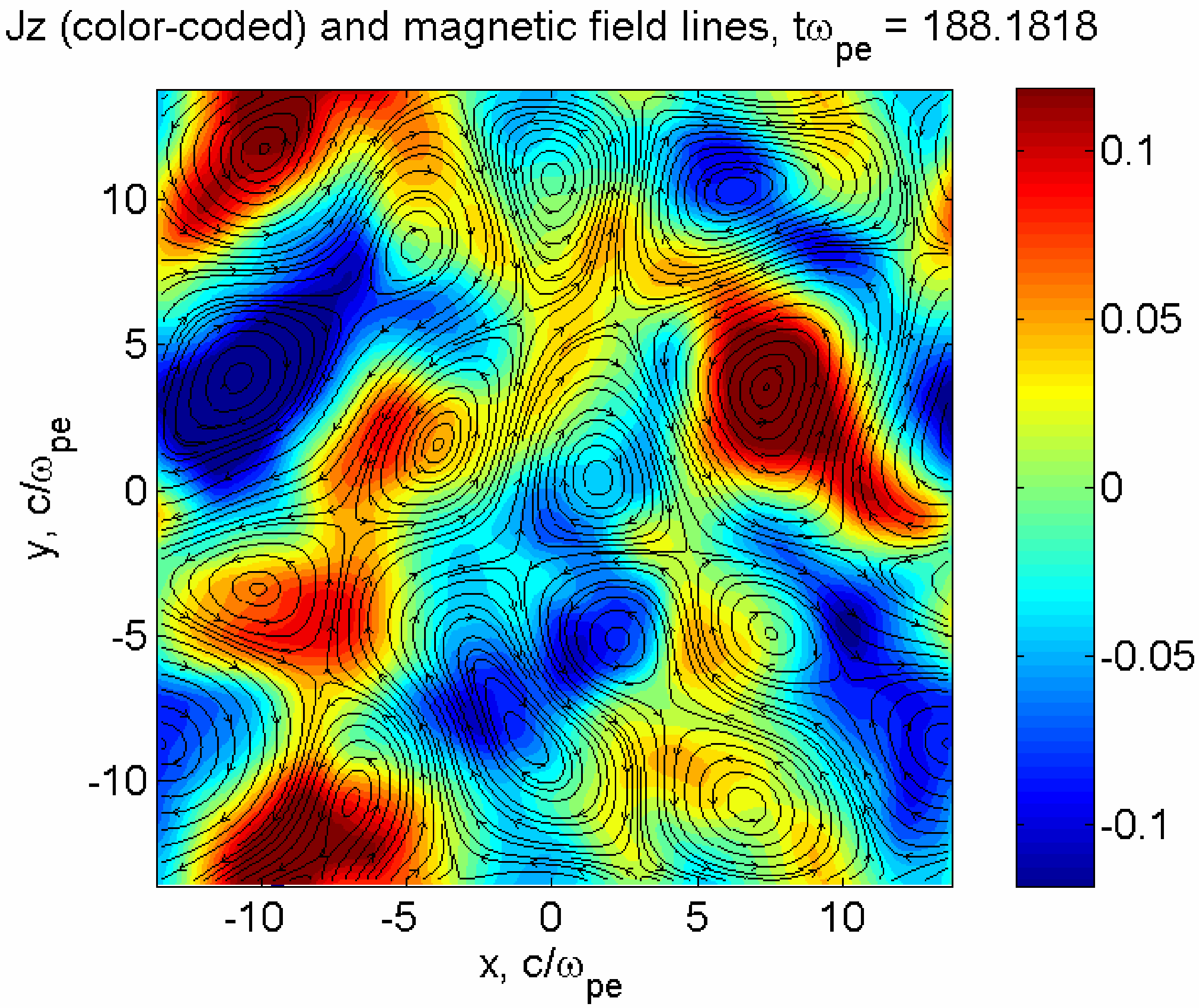
$$w_{i+1/2}^{n+1} = \frac{1}{2}\left(\overline{w}_i^n + \overline{w}_{i+1}^n\right) + \overbrace{\frac{1}{8}\left(w^x{}_i - w^x{}_{i+1}\right)}^{\text{Slope limiter}} - \frac{\lambda}{2}\left[F(w_{i+1}^{n+1/2}) - F(w_i^{n+1/2})\right]$$

Test problems: Weibel instability

- The Weibel instability is driven in a collisionless plasma by the anisotropy of the particle velocity distribution function of the plasma
  - Shocks
  - Strong temperature gradient
- Magnetic fields are generated so that the distribution function becomes isotropic

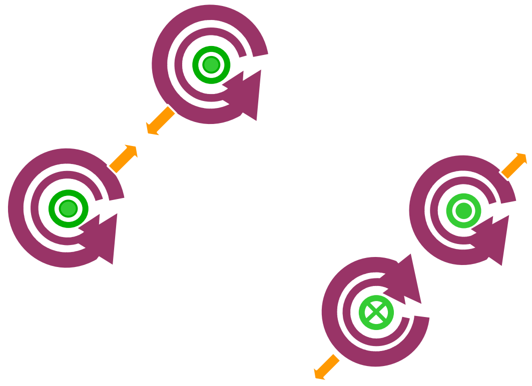


Initial electron temperature is anisotropy Tzz = 10Txx, ions are isotropic. Ion mass is Mi = 25Me. The simulation is performed on a 2D domain (Nx = Ny = 128). Periodic boundary conditions are adopted in both coordinate directions.


Jz (color-coded) and magnetic field lines, $t\omega_{pe}$ = 188.1818
y, c/ω_{pe}
x, c/ω_{pe}
10
5
0
-5
-10
-10
-5
0
5
10
0.1
0.05
0
-0.05
-0.1

Comments on Weibel instability development

- **The process of instability development is accompanied by creation of localised current sheets, sustained by self-consistent magnetic fields. Currents with the same direction are attracted because of their magnetic field.**



  - **Currents and magnetic fields increase through merger of currents due to magnetic field lines reconnection. This leads to decrease of temperature anisotropy.**

Multi fluid simulations
Summary

- Solve coupled system of fluid and Maxwell equations.
- Uses first 10 moments of 6D-distribution functions.
- Written as first order in time system.
- Flux-conservative part + Source-term.
- Based on Lax-Wendroff scheme.
- Slope-limiter to avoid spurious oscillations near strong gradients.
- Tested with Weibel instability in anisotropic plasma.

I am grateful to all people who helped me to prepare this lecture by providing material, discussions and checking lecture notes and exercises:

- Nina Elkina
- Julia Thalmann
- Tilaye Tadesse
- Elena Kronberg
- Many unknown authors of Wikipedia and other online sources.

For this lecture I took material from

- Wikipedia and links from Wikipedia
- Numerical recipes in C, Book and http://www.fizyka.umk.pl/nrbook/bookcpdf.html
- Lecture notes *Computational Methods in Astrophysics* http://compschoolsolaire2008.tp1.ruhr-uni-bochum.de/
- Presentation/Paper from Nina Elkina
- MHD-equations in conservative form: http://www.lsw.uni-heidelberg.de/users/sbrinkma/seminar051102.pdf


THE
END