Helioseismology

Laurent Gizon (lecturer)

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These are the lecture notes of a lecture given at the University of Göttingen and the Max Planck Institute for Solar System Research in Katlenburg-Lindau, respectively.
The lecture was given by Dr. Laurent Gizon. These notes are summaries of the lecture, written down by students which were attending the lecture.
The authors of each single chapter are in detail:

Chapter 2:	Nazaret Bello González, Julián Blanco Rodríguez, Bruno Sánchez-Andrade Nuno, Thorsten Stahn, Danica Tothova
Chapter 3:	Nazaret Bello González, Julián Blanco Rodríguez, Nilda Oklay, Bruno Sánchez-Andrade Nuno, Thorsten Stahn, Danica Tothova, Esa Villenius, Jean-Baptiste Vincent
Chapter 4:	Khalil Daiffallah, Kristian Hallgren, Emre Isik, Christian Koch, Cornelia Martinecz, Martin Meling, Silvia Protopapa, Pedro Russo, Jean Santos, Clementina Sasso, Sofi Spjuth, Julia Thalmann, Cecilia Tubiana, Lotfi Yelles
Chapter 5:	Christian Koch

For further information, additional material, and useful links you may visit the following website: Helioseismology lecture

Contents

Chapter 1

Short survey of helioseismology and asteroseismology aims and results

Here is a list of some topics in solar (and stellar) physics which may be investigated with the tool of helioseismology:

- What is the mechanism of the solar cycle?
- Dynamo theory: How does motions in the sun generate the magnetic field of the sun
- Large scale flows (e.g. meridional flows), convective flows
- The internal magnetic field of the sun
- Active regions: (detailed) structure, emergence, evolution
- Space weather
- Basic physics: neutrinos, G, micro physics (equation of state), etc.

For a more detailed overview about the aims of helio- and asteroseimology as well as some of its results including some nice pictures and movies, have a look at the Power Point Presentation “Introduction” on the website of the lecture (http://www.mps.mpg.de/projects/seismo/Helioseismology.html).

Chapter 2

Observed properties of stars

2.1 Isolated spherical stars

In this section we assume an isolated, non magnetic, non rotating star. In that case it has a perfect spherical shape. If this star has a companion, forming a binary system, or any other massive body around, this could create deformations, and tidal forces, which we won't take into consideration. We will also assume that there is no interstellar medium in the surroundings, so that our line of sight is unperturbed. The assumption of neglecting rotation and magnetic fiels means neglecting flattening of the polar regions due to the centrifugal forces, and stresses inside the star due to magnetic fields.

2.2 Factors influencing observed properties of stars

Many characteristic properties of a star (such as structure, evolution and lifetime) are determined by its initial conditions, that is basically its initial mass, M_s, and its initial chemical composition.
During its evolution, a star passes significant changes in some of its properties (e.g. Radius, Luminosity, Temperature, rotational speed), so that the stellar age becomes another important factor influencing the observed properties.
Finally, what we observe from a star, will be influenced by the distance between the star and the observer and the interstellar medium that may be block or emit in some spectral regions

2.3 Radiation from the star

The radiation which is emitted by a star can be analyzed *quantitatively*, as the integrated flux over a given spectral region, or *qualitatively* studying the shape of the spectral lines.
Since, usually, most of the light comes from the region where we almost have thermodynamic equilibrium, the energy distribution of the star can be approximated

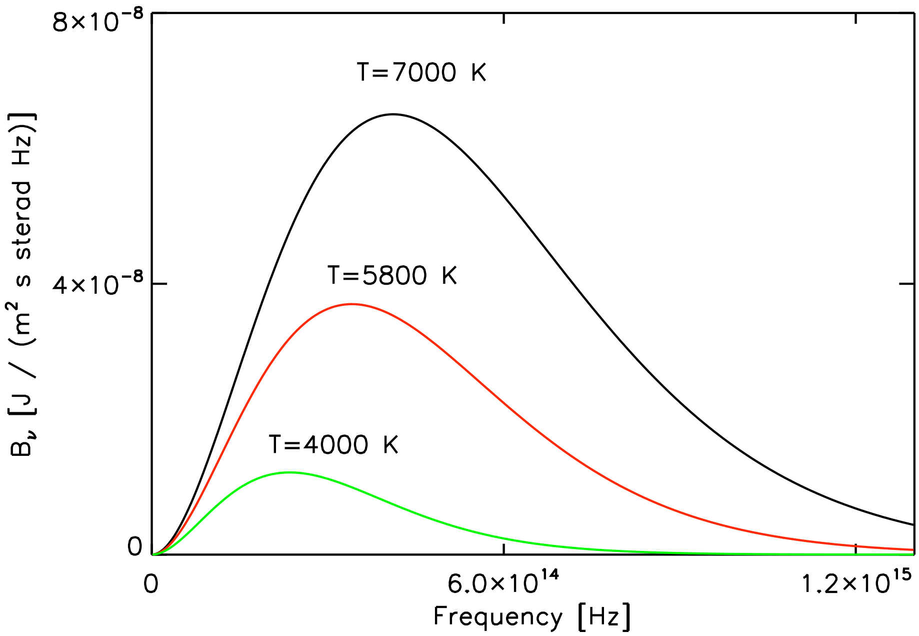

Figure 2.1: Black body spectra for different temperatures.

by a black body, which follows the Planck law:

$$I_\nu = B_\nu(T_s) = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{k_B T_s}\right) - 1\right]^{-1} \tag{2.1}$$

In the prior equation, h, c and k_B denote, respectively, the Planck's constant, the speed of light and the Boltzmann constant. By considering a star as a black body, one may describe the Intensity I_ν of the emitted light of a star at a certain frequency ν by a Planck function, $B_\nu(T_s)$, which solely depends on the surface temperature T_s. Figure 2.1 shows Intensity spectra for black bodies with different temperatures.

In reality, the black body spectrum of star is superimposed by spectral lines from photons with higher or lower energy than the ones of the black body background approximation. Those photons come from regions where we might not have such equilibrium, or a different temperature, leading to abortions or emission lines. Those lines will be mostly seen as absorption lines, due to the temperature gradient. They not only reveal the chemical composition of the star, but may also be used to measure stellar oscillations trough their periodic doppler shifts.

The Luminosity of a star is the total amount of energy radiated at any direction per unit time (measured in W or J/s) and it is given by

$$L_s = 4\pi R_s^2 \sigma T_s^4, \tag{2.2}$$

where R_s is the stellar radius and σ is the Stefan-Boltzmann constant. Since a star is not strictly a black body, one cannot express the luminosity in terms of a surface temperature. Therefore, one may introduce an effective Temperature, $T_{\rm eff}$, which is the temperature of a black body which has the same luminosity as the star and is defined according to the Stefan-Boltzmann law, $F = \sigma T_{\rm eff}^4$. Thus, equation (2.2) becomes

$$L_s = 4\pi R_s^2 \sigma T_{\rm eff}^4. \tag{2.3}$$

2.4 Mass M_s

There are just few different ways to determine the mass of a star, of which a few examples could be:

1. Apply Kepler's law to binary systems, or for example in the case of the solar system, the sun and planets.

2. With asteroseismology we have an independent method to determine stellar masses even of single stars, using the best fit model that matches the observations.

2.5 Stellar radius

Stellar radii may be measured

- by occultation of a star with an object without atmosphere, e.g. the moon,

- by direct imaging, until now only possible trough interferometry,

- by means of asteroseismology which provides the most precise radii measurements.

Chapter 3

Dynamical structure of stars

3.1 Hydrostatic equilibrium

Consider a thin spherical mass element with thickness δr and surface δs at a radius r in the star. The gravitational force

$$F_G = \frac{G[\rho(r)\delta s \delta r]M(r)}{r^2} \tag{3.1}$$

acts on the mass element towards the stellar center. This gravitational force has to be compensated by the difference of pressure force acting acting at radii r and $r + \delta r$:

$$[P(r + \delta r) - P(r)]\delta s = -F_G \tag{3.2}$$

Regarding that $P(r + \delta r) - P(r) = \frac{\partial P}{\partial r}\delta r$, equation (3.2) finally becomes the hydrostatic equation describing the solar structure:

$$\frac{\partial P}{\partial r} = -\frac{G\rho(r)M(r)}{r^2} \tag{3.3}$$

Note that in the equation above, $\rho(r)$ donotes the local density at radius r while $M(r)$ denotes an integral measure of the mass from the center up to the radius r.

3.2 Validity of this approximation

In order to proof the validity of the hydrostatic equilibrium, we may write the equation of motion (including only forces acting in radial direction) as

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \partial_r v_r\right) = -\frac{\partial P}{\partial r} + \rho\frac{\partial \phi}{\partial r}, \tag{3.4}$$

where the momentum of a mass element on left hand side of is generated by pressure forces and the gravitational force on right hand side.
In general, the gravitational field inside the star can be described in terms of a

gravitational potential ϕ which is the solution of a Poisson equation (here in spherical symmetry):

$$\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\frac{\partial\phi}{\partial r}\right) = -4\pi G\rho \tag{3.5}$$

Integration of equation (3.5) leads to an expression for $\frac{\partial\phi}{\partial r}$:

$$\int_0^r \partial_{r'}\left(r'^2\frac{\partial\phi}{\partial r'}\right)dr' = -\int_0^r 4\pi G\rho r'^2 dr \tag{3.6}$$

$$r^2\frac{\partial\phi}{\partial r} = -GM(r) \tag{3.7}$$

$$\frac{\partial\phi}{\partial r} = \frac{-GM(r)}{r^2} = g(r) \tag{3.8}$$

Thus, equation (3.4) becomes

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\partial_r v_r\right) = -\frac{\partial P}{\partial r} + \frac{\rho GM(r)}{r^2} \tag{3.9}$$

To proof the validity of the hydrostatic equilibrium, let us suppose

$$\rho\frac{dv_r}{dt} = -\epsilon\frac{\rho GM(r)}{r^2}, \qquad \text{with: } \epsilon \ll 1 \tag{3.10}$$

As the most simple approximation, we can say the time for a substantial collapse of the star is given by the kinematic equation of the free fall in a gravitation field:

$$r = \tfrac{1}{2}g(r)t^2 = \tfrac{1}{2}\epsilon\tfrac{GM(r)}{r^2}t^2 \tag{3.11}$$

$$\Rightarrow t = \sqrt{\tfrac{1}{\epsilon}\tfrac{2r^3}{GM(r)}} \tag{3.12}$$

For the sun, we use $r = R_\odot$ and $M(r) = M(R_\odot) = M_\odot$ and obtain a collapse time of $t = 2 \cdot 10^3 \cdot \epsilon^{-1/2}$sec. Assuming a minimal solar age of $t > 4.4 \cdot 10^9$yr (according to the age of the oldest rocks on earth), one gets $\epsilon \sim 10^{-28}$, showing that te assumption of hydrostatic equilibrium is very good.

Finally, we can summarise the first two equations describing the inner structure of a star:

1. Mass conservation:

$$\frac{dM(r)}{dr} = 4\pi\rho(r)r^2 \tag{3.13}$$

2. Hydrostatic equilibrium:

$$\frac{\partial P}{\partial r} = -\frac{G\rho(r)M(r)}{r^2} \tag{3.14}$$

3.3 Equation of state

With an assumption of a star is an ideal gas:

$$P = P_{gas} + P_{radiation} \tag{3.15}$$

$$P_{gas} = nkT = \frac{\rho(r)}{\bar{m}} kT \tag{3.16}$$

Mean molecular weight: $\bar{m} = \mu m_H$

$$P_{gas} = nkT = \frac{\rho(r)}{m} kT \tag{3.17}$$

$$P_{gas} = \rho(r) \frac{\mu m_H}{kT} \tag{3.18}$$

$$\tag{3.19}$$

Gas constant $R = k/m_H$

$$P_{gas} = \rho(r) \frac{\mu}{R} T P_{rad} = 1/3 a T^4 \tag{3.20}$$

Equation of State:

$$P = \frac{\rho R T}{\mu} + \frac{1}{3} a T^4 \tag{3.21}$$

3.4 The virial Theorem

$$-\Omega = 3 \int_0^V P dV \tag{3.22}$$

Ω is the negative gravitational energy of the star; V volume of a sphere of radius r

$$3 \int_0^V P dV = 3[PV]_c^s - 3 \int_c^s V dP \tag{3.23}$$

$$= -3 \int_c^s V dP \tag{3.24}$$

$$= 4\pi \int_c^s \frac{GM(r) r^3 \rho(r)}{r^2} dr \tag{3.25}$$

$$3 \int_c^s P dV = \int_c^s \frac{GM(r)}{r} dM \tag{3.26}$$

$$\tag{3.27}$$

$\delta\Omega$ is the work required to bring δM from infinity to the sphere of radius r.

$$\delta\Omega = \int_{\infty}^{r} \frac{GM(r)\delta M}{x^2}dx = -\frac{GM(r)\delta M}{r} \quad (3.28)$$

$$\Omega = -\int_{c}^{s} \frac{GM(r)dM}{r} \quad (3.29)$$

$$-\Omega = \int_{c}^{s} \frac{GM(r)dM}{r} = 3\int_{c}^{s} PdV \quad (3.30)$$

Theorem:

$$-\Omega = \int_{c}^{s} \frac{GM(r)dM}{r} > \int_{c}^{s} \frac{GM(r)dM}{r_s} \quad (3.31)$$

$$\int_{c}^{s} \frac{GM(r)dM}{r_s} = \frac{GM_s^2}{2r_s} \quad (3.32)$$

$$-\Omega > \frac{GM_s^2}{2r_s} \quad (3.33)$$

$$(3.34)$$

Internal energy per unit volume is $u = \frac{1}{\gamma-1}P$
γ is the ratio of specific heat: $\gamma = \frac{c_p}{c_v}$

$$P = (\gamma-1)U \quad (3.35)$$

$$-\Omega = 3\int(\gamma-1)UdV = 3(\gamma-1)U \quad (3.36)$$

$$-\Omega = 3(\gamma-1)U \quad (3.37)$$

$$(3.38)$$

The total energy of a star would be

$$E = \Omega + U = (4-3\gamma)U \quad (3.39)$$

With this equation we can see where the star is stable or unstable:
E < 0 stable $\rightarrow \gamma > 4/3$

E > 0 unstable $\rightarrow \gamma < 4/3$

For monoatomic gas, $\gamma = 5/3$

$$\gamma = 5/3 \rightarrow \Omega + U = -U \quad (3.40)$$

$$\gamma + 2U = 0 \quad (3.41)$$

3.5 Another derivation

We can also derivate this theorem from Euler's equation of fluid dynamics.

$$\ddot{\vec{r}} = -\frac{1}{\rho}\overrightarrow{grad}P + \vec{F}_{grav} \tag{3.42}$$

Then we multiply by $\vec{r}$ and integrate on the whole mass M

$$\int_M \vec{r}.\ddot{\vec{r}}dm = -\int_M \frac{1}{\rho}\vec{r}.\overrightarrow{grad}Pdm + \int_M \vec{r}.\vec{f}_{grav}dm \tag{3.43}$$

We will now express separately the three components of this equation

(i)

$$\begin{aligned}
\int_M \vec{r}.\ddot{\vec{r}}dm &= \int_M \left(\frac{d}{dt}(\vec{r}.\dot{\vec{r}}) - \dot{\vec{r}}^2\right)dm \\
&= \int_M \left(\frac{1}{2}\frac{d^2}{dt^2}\vec{r}^2 - \dot{\vec{r}}^2\right)dm
\end{aligned}$$

Here one can recognize the formulas of kinetic energy and momentum of inertia defined by

$$I = \int_M r^2 dm$$

$$KE = \frac{1}{2}\int_M \dot{r}^2 dm$$

Finally,

$$\int_M \vec{r}.\ddot{\vec{r}}dm = \frac{1}{2}\frac{d^2 I}{dt^2} - 2KE \tag{3.44}$$

(ii)

$$\begin{aligned}
\int_M \frac{1}{\rho}\vec{r}.\overrightarrow{grad}Pdm &= \int_V \vec{r}.\overrightarrow{grad}PdV \\
&= \int_V div(\vec{r}P)dV - \int_V Pdiv(\vec{r})dV \\
&= \int_V div(\vec{r}P)dV - 3\int_V PdV \\
&= \int_S P\vec{r}dS - 3\int_V PdV
\end{aligned}$$

We assume P = 0 at the surface of the star, so finally

$$\int_M \frac{1}{\rho}\vec{r}.\overrightarrow{grad}P dm = -3\int_V P dV \tag{3.45}$$

(iii)

$$\int_M \vec{r}.\vec{f}_{grav} dm = \Omega = \text{Potential Energy of Gravitation} \tag{3.46}$$

At the end, we can now use 3.44, 3.45, 3.46 in 3.43, which brings us to another expression of the virial theorem.

$$\frac{1}{2}\frac{d^2 I}{dt^2} - 2KE = 3\int_V P dV + \Omega \tag{3.47}$$

$$\frac{1}{2}\ddot{I} = 2\text{KE} + 3\left(3\gamma - 1\right)U + \Omega \tag{3.48}$$

The gravitational energy of an isolated star is

$$-\Omega = \int \frac{GM\left(r\right)}{r} dM. \tag{3.49}$$

If the structure of the star is homologous, fixed functions $M\left(\frac{r}{r_s}\right)$ and $\rho\left(\frac{r}{r_s}\right)$ can be applied. In other words, the functional form remains the same when the size of a star is scaled. Making the following change of variables

$$M\left(r\right) = M_s f_1\left(\frac{r}{r_s}\right) \tag{3.50}$$

$$\frac{dM}{dr}dr = dM = \frac{M_s}{r_s} f_1'\left(\frac{r}{r_s}\right) dr \tag{3.51}$$

gives

$$-\Omega = \frac{GM_s^2}{r_s} q, \tag{3.52}$$

where q is the constant number

$$q = \int_0^1 \frac{f_1\left(x\right) f_1'\left(x\right)}{x} dx. \tag{3.53}$$

Now

$$I = \int_{\text{star}} r^2 dM \tag{3.54}$$

$$I = sM_s r_s^2, \tag{3.55}$$

where s is a constant.

$$r_s = r_0 + \varepsilon \Delta r(t) \tag{3.56}$$

$$\frac{1}{2}\ddot{I} = (3\gamma - 1) E + (4 - 3\gamma) \Omega \tag{3.57}$$

For a static case, when a star is in equilibrium, this becomes

$$0 = (3\gamma - 1) E + (4 - 3\gamma) \Omega_0. \tag{3.58}$$

$$\begin{aligned}
\frac{1}{2}\ddot{I} &= \frac{1}{2} s M_s \partial_t^2 (r_0 + \varepsilon \Delta r)^2 && (3.59)\\
&= \frac{1}{2} s M_s \partial_t \left[(r_0 + \varepsilon \Delta r) 2\varepsilon \Delta \dot{r} \right] && (3.60)\\
&= \frac{1}{2} s M_s 2 r_0 \varepsilon \Delta \ddot{r} && (3.61)\\
& && (3.62)
\end{aligned}$$

$$\begin{aligned}
\Omega &= -q \frac{G M_s^2}{r_s} && (3.63)\\
&= -q \frac{G M_s^2}{r_0} \left(1 + \varepsilon \frac{\Delta r}{r_0}\right)^{-1} && (3.64)\\
&= \Omega_0 + q \frac{G M_s^2}{r_0^2} \Delta r \varepsilon && (3.65)\\
&= \Omega_0 \left(1 - \varepsilon \frac{\Delta r}{r_0}\right) && (3.66)\\
\Rightarrow s M r_0 \varepsilon \Delta \ddot{r} &= (3\gamma - 4) \Omega_0 + (4 - 3\gamma) \Omega_0 \left(1 - \varepsilon \frac{\Delta r}{r_0}\right) && (3.67)\\
s M_s r_0 \Delta \ddot{r} &= -(4 - 3\gamma) \Omega_0 \frac{\Delta r}{r_0} && (3.68)
\end{aligned}$$

Case $\gamma > \frac{4}{3}$, stable oscillations

$$\Delta \ddot{r} = C \Delta r, \tag{3.69}$$

where $C > 0$ is a constant:

$$C = -\frac{\Omega_0 (4 - 3\gamma)}{s r_0^2 M_s}. \tag{3.70}$$

Case $\gamma < \frac{4}{3}$

$$C = -\frac{\Omega_0 (4 - 3\gamma)}{I q}. \tag{3.71}$$

Thus, $\Delta r = A \exp i\omega t$, where $-\omega^2 = -(3\gamma - 4)\frac{-\Omega_0}{I_0}$, where $\omega = \frac{2\pi}{T}$, T is the pulsation period. Substituting ω and solving for T gives

$$\left(\frac{2\pi}{T}\right)^2 = (3\gamma - 4)\frac{-\Omega_0}{I_0} \tag{3.72}$$

$$T = \frac{2\pi}{\sqrt{(3\gamma - 4)\left(\frac{-\Omega_0}{I_0}\right)}} \tag{3.73}$$

$$T = 2\pi\sqrt{\frac{s r_s^3}{(3\gamma - 4)\, qGM_s}}, \tag{3.74}$$

where s and q are constant for all stars. It may be observed that the mean density is proportional to $\frac{M_s}{r_s^3}$ and the period to $\frac{1}{\sqrt{\rho}}$. For the Sun the period of the fundamental radial oscillation mode is of the magnitude of hours.

Chapter 4

Linear Stellar Oscillations

4.1 Eulerian and Lagrangian perturbations

The choice of independant variables differentiates the Eulerian and Lagrangian fluid descriptions. The position vector $\boldsymbol{r}$ and time t are the independent variables in an Eulerian fluid and any perturbation of the quantity Q is writen as $Q' = Q'(\boldsymbol{r}, t)$. This description is completly general since we suppose that the value of a variable at any point is uncorrelated with the value at a neighboring point. In contraste, the Lagrangian description divides a fluid into tiny parcels. The independent variables are the time t and the displacement vector $\xi = \boldsymbol{r} - \boldsymbol{r}_0$ which are associated with parcels fluid, not points in space. The Lagrangian perturbation is denoted $\delta Q(\xi, t)$. In terms of time derivatives in the case of Eulerian description, we write:

$$\partial/\partial t \equiv (\partial/\partial t)_{\boldsymbol{r}} \tag{4.1}$$

derivative at fixed $\boldsymbol{r}$.

In the case of Lagrangian description, we can write :

$$D_0/Dt = (\partial/\partial t)_{\boldsymbol{r}} + (\boldsymbol{v}_0 \cdot \nabla) \tag{4.2}$$

derivative at fixed $\boldsymbol{r}_0$ and comoving with to the backround flow $\boldsymbol{v}_0$.

Velocity perturbation:

The displacement vector is defined as $\xi = \boldsymbol{r} - \boldsymbol{r}_0$. The first order Lagrangian variation of the vector $\boldsymbol{v}$ can writen as :

$$\delta \boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r} + \boldsymbol{\xi}, t) - \boldsymbol{v}_0(\boldsymbol{r}, t) = \boldsymbol{v}(\boldsymbol{r}, t) + (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{v} - \boldsymbol{v}_0(\boldsymbol{r}, t) \tag{4.3}$$

we have also $\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{v}_0(\boldsymbol{r}, t) + \boldsymbol{v}'(\boldsymbol{r}, t)$ where $\boldsymbol{v}'(\boldsymbol{r}, t)$ is the Eulerien perturbation of the velocity. The equation 4.3 become :

$$\delta \boldsymbol{v} = \boldsymbol{v}'(\boldsymbol{r}, t) + (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{v}_0 \tag{4.4}$$

From the definition of $\boldsymbol{\xi}$ we have also:

$$\delta \boldsymbol{v} = \frac{D_0 \boldsymbol{\xi}}{Dt} = \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{v}_0 \cdot \nabla)\boldsymbol{\xi} \tag{4.5}$$

then we obtain :

$$\boldsymbol{v}' = \delta \boldsymbol{v} - (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{v}_0 = \frac{D_0 \boldsymbol{\xi}}{Dt} - (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{v}_0 \tag{4.6}$$

4.1.1 Equations of the fluid

Mass conservation equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{4.7}$$

Momentum equation:

$$\rho D_t \boldsymbol{v} = -\nabla p - \rho \nabla \phi \tag{4.8}$$

where ϕ is the gravitational potential

Poisson equation:

$$\triangle \phi = 4\pi G \rho \tag{4.9}$$

where G is the constant of gravity

Energy equation:

$$\rho T D_t S = \rho \epsilon - \nabla \cdot \boldsymbol{F} \tag{4.10}$$

where S is the specific entropy, $\rho\epsilon$ is the internal energy density, $\boldsymbol{F}$ is the energy flux and T is the temperature.

4.1.2 Equations of linear oscillations :

At the equilibrium state, we have $\partial_t = 0$ and the partial derivative in space vanishes also, we have also $\boldsymbol{v}_0 = 0$, we can write the fluid equations in this case as :

$$0 = -\nabla p_0 - \rho_0 \nabla \phi_0 \tag{4.11}$$

$$\triangle \phi_0 = 4\pi G \rho_0 \tag{4.12}$$

$$0 = \rho_0 \epsilon_0 - \nabla \cdot \boldsymbol{F}_0 \tag{4.13}$$

Perturbed state: Now we perturbe the pressure, the potential of the gravity, the density and the velocity with Eulerian perturbations, we replace all theses quantities in the fluid equations. After the linearisation we obtain the equations of linear oscillations:

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\nabla p' - \rho_0 \nabla \phi' - \rho' \nabla \phi_0 \tag{4.14}$$

$$\partial_t(\rho' + \nabla \cdot (\rho_0 \boldsymbol{\xi})) = 0 \tag{4.15}$$

$$\triangle \phi' = 4\pi G \rho' \tag{4.16}$$

$$\rho_0 T_0 \partial_t(\delta S) = (\rho_0 \epsilon)' - \nabla \cdot \boldsymbol{F}' \tag{4.17}$$

4.2 Eigenvalue problem of linear oscillations

4.2.1 Eigenvalue problem

The basic equations of linear stellar oscillations (see previous section), describe linear adiabatic oscillations and can be solved for particular boundary conditions. Of course, these boundary conditions are satisfied for a set of special values of the frequency ω – the eingenvalues.

4.2.2 Boundary conditions

Sun center: As $r \to 0$, c and $\rho \to$ to a constant value, so one can write

$$g = \frac{GM_r}{r^2} \approx \frac{4\pi}{3} r^3 G \frac{\rho_c}{r^2} \to 0.$$

being ρ_c the value of the density at the Suns center. Furthermore, for the Schwarzschild discriminant A:

$$A \to 0 \;\text{ and }\; \mathrm{L}_1^2 \to \frac{1}{\mathrm{r}^2}. \tag{4.18}$$

After this assumptions, the simplified eigenvalue problem appears as

$$\frac{d}{dr}\left(r^2 \xi_r\right) - \frac{l(l+1)}{\omega^2}\left(\phi' + \frac{p'}{\rho}\right) \simeq 0, \tag{4.19}$$

$$\frac{1}{\rho}\frac{dp'}{dr} - \omega^2 \xi_r + \frac{d\phi'}{dr} \simeq 0, \tag{4.20}$$

$$\frac{d}{dr}\left(r^2 \frac{d\phi'}{dr}\right) - l(l+1)\phi' \simeq 0. \tag{4.21}$$

Equating the different forms of the Poisson equation, one obtains

$$\left(r\frac{d}{dr} - l\right)\left(r\frac{d}{dr} + l + 1\right)\phi' \simeq 0. \tag{4.22}$$

This form of the Poisson equation now has two solutions, which are

$$r\frac{d}{dr} - l \quad = \quad 0 \quad \rightarrow \quad \phi' \sim r^l, \tag{4.23}$$

$$r\frac{d}{dr} + l + 1 \quad = \quad 0 \quad \rightarrow \quad \phi' \sim r^{-(l+1)}, \tag{4.24}$$

where the latter is not a regular solution, since it goes to infinity for $r \rightarrow 0$. Thus, we have (corresponding to 4.23)

$$r\frac{d\phi'}{dr} = l\phi', \tag{4.25}$$

and, combining 4.19, 4.20, and 4.23, one obtains for the radial component of the displacement

$$\xi_r \simeq \frac{1}{\omega^2 r^2}\left(\frac{p'}{\rho} + \phi'\right) \text{ for r} \rightarrow 0. \tag{4.26}$$

Solar surface: At the solar surface $r \rightarrow R_\odot$ the boundary conditions develop as follows (Free surface boundary condition). From $\delta p = 0$:

$$\begin{aligned} p' + \xi \cdot \nabla p &= p' - \rho g \xi_r = 0, \\ p' &= \rho g \xi_r, \end{aligned} \tag{4.27}$$

As a second surface boundary condition, which only lasts in the Cowling approximation, we have $\rho \rightarrow 0$, resulting in

$$\xi_r = \frac{\omega^2 R_\odot}{g}\xi_h = \frac{\omega^2 R_\odot^3}{GM_\odot}\xi_h = (\omega\tau_{dyn})^2\xi_h, \tag{4.28}$$

where τ_{dyn} denotes the dynamical time scale.

Our problem is still degenerate in m, since we assumed not to have effects of the magnetic field, rotation, and so on. This means, that if we evaluate the same eigenfunctions for different m, we will obtain the same eigenvalues.

4.2.3 Orthogonality of eigenfunctions

Now, the eigenvalue problem can be written as

$$\mathcal{L}[\vec{\xi}] = \omega^2 \rho \vec{\xi}, \tag{4.29}$$

which is the vector form of it and where $\mathcal{L}$ denotes the linear, second-order differential operator. The solution of the problem is of the form

$$\mathcal{L}[\vec{\xi}_{nlm}] = \omega^2_{nlm}\, \rho\, \vec{\xi}_{nlm}, \tag{4.30}$$

i.e. that for each (l, m) one can find a number of solutions labeled with n. Then, n gives the number of modes of ξ_r in the radial direction. Now, one searches functions which satisfy the boundary conditions at the center and the surface of the Sun, at the same time. One finds,

$$\begin{aligned} \int dV \vec{\xi}^* \cdot \mathcal{L}(\vec{\xi}') = \\ \int dV \Big[-\xi^* \cdot \nabla(\rho c^2 \nabla \cdot \xi') - \xi^* \cdot \nabla(\nabla\rho \cdot \xi') + \frac{\nabla p \cdot \xi^*}{\rho} \nabla \cdot (\rho \xi') \\ + \rho \xi^* \cdot \nabla \Big[G \int dV' \frac{\nabla'[\rho(r')\xi'(r')]}{|r - r'|} \Big] \Big]. \end{aligned} \tag{4.31}$$

This is then simplified by the 'integration by parts', obtaining

$$\begin{aligned} &\int dV \left[\rho c^2 (\nabla \cdot \xi^*)(\nabla \cdot \xi') + (\nabla \cdot \xi^*)(\nabla p \cdot \xi') \right] \\ &+ \int dV \left[(\nabla p \cdot \xi^*)(\nabla \cdot \xi') + \frac{(\nabla p \cdot \xi^*)(\nabla \rho \cdot \xi')}{\rho} \right] \\ &- G \int dV \int dV' \frac{\nabla[\rho(r)\xi^*(r)] \nabla'[\rho(r')\xi'(r)]}{|r - r'|}. \end{aligned} \tag{4.32}$$

The main advantage of the form of 4.32 is that one can show now the invariant behaviour of this equation. It also holds, if one swaps ξ and ξ'. Thus, it is

$$\int dV \xi^* \cdot \mathcal{L}(\xi') = \int dV \xi'^* \cdot \mathcal{L}(\xi) = \int dV [\mathcal{L}(\xi)]^* \cdot \xi', \tag{4.33}$$

then $\mathcal{L}$ is self-adjoint. Furthermore, one obtains for (n, l, m), after using 4.29 the relation

$$\omega^2_{n'l'm'} \int dV \xi^*_{nlm} \cdot \xi_{n'l'm'} \rho = \omega^{*2}_{nlm} \int dV \xi^*_{nlm} \cdot \xi_{n'l'm'} \rho. \tag{4.34}$$

It can be shown that if $(n, l, m) = (n', l', m')$ then since the two integrals in 4.34 are equal:

$$\omega^2_{n'l'm'} = \omega^{*2}_{nlm}, \tag{4.35}$$

frequencies are real.

This means, that unless some modes are degenerate, a given mode corresponds to a given frequency. Last, the orthogonality of the eigenvectors $\vec{\xi}_{nlm}$ is given by

$$\int dV \xi_{nlm} \cdot \xi_{n'l'm'} \rho = \delta_{nn'} \delta_{ll'} \delta_{mm'} \int dV ||\xi_{nlm}||^2 \rho. \tag{4.36}$$

4.2.4 Variational principle

Note, that in the following the displacement vector will simply be denoted by writing ξ instead of $\vec{\xi}$. To obtain a very good estimate of frequencies when one is not able to solve the eigenvalue problem, one makes use of the variational principle, which can be written as

$$\int dV \xi^* \cdot \mathcal{L}(\xi) = \omega^2 \int dV ||\xi||^2 \rho. \tag{4.37}$$

Effects of change of the displacement vector are represented by the substitution $\xi \to \xi + \Delta\xi$. Now, one looks at the corresponding changes of the frequency $\omega^2 \to \omega^2 + (\Delta\omega)^2$ and checks if 4.37 holds,

$$\begin{aligned} \int dV \left[\Delta\xi^* \cdot \mathcal{L}(\xi) + \xi^* \cdot \mathcal{L}(\Delta\xi)\right] = \\ (\Delta\omega)^2 \int dV ||\xi||^2 \rho + \omega^2 \int dV (\xi^* \cdot \Delta\xi + \xi \cdot \Delta\xi^*) \rho, \end{aligned} \tag{4.38}$$

which represent the first-order changes that remain. Furthermore, since $\mathcal{L}$ is self-adjoint:

$$\begin{aligned} (\Delta\omega)^2 \int dV ||\xi||^2 \rho = \\ \int dV \left[\Delta\xi^* \cdot \mathcal{L}(\xi) + (\mathcal{L}(\xi^*))^* \cdot \Delta\xi - \omega^2 \rho \xi^* \cdot \Delta\xi - \omega^2 \rho \xi \cdot \Delta\xi^*\right] = \\ 2\Re \int dV \Delta\xi^* \cdot \left[\mathcal{L}(\xi) - \omega^2 \rho \xi\right]. \end{aligned} \tag{4.39}$$

This implies, that $(\Delta\omega)^2 \to 0$, i.e. ω^2 is stationary with respect to a change $\Delta\xi$ in ξ. In other words, this means that

$$(\Delta\omega)^2 = 0 \text{ (for an arbitrary } \Delta\xi) \iff \mathcal{L}(\xi) = \omega^2\rho\xi.$$

We show the implication $\Longleftarrow$.

This forms the basis of the so-called 'Rayleigh-Ritz formula' which is

$$\omega^2 = \frac{\int dV(\xi + \Delta\xi)^* \cdot \mathcal{L}[\xi + \Delta\xi]}{\int dV\|\xi + \Delta\xi\|^2\rho} + o(|\Delta\xi|^2). \qquad (4.40)$$

So, the important thing concerning the variational principle is, that if one makes a small variation inside the system one is still able to get a good approximation of the frequency. In other words, if one assumes a small change $\xi + \Delta\xi$ in ξ, one will still get the right ω.

4.3 Linear perturbation theory

4.4 Asymptotic description of p-mode frequencies

4.4.1 Duvall's law

If the wavelength is smaller than the variations in the local medium, the accoustic-wave dipersion relation takes in an important role:

$$\omega^2 = c^2 \mid \mathbf{k} \mid^2 = c^2(k_r^2 + k_h^2) = c^2\left(k_r^2 + \frac{l(l+1)}{r^2}\right), \qquad (4.41)$$

where

$$k_r = \left[\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2}\right]. \qquad (4.42)$$

k_r has to fit the standing wave condition. Further a surface- induced phase shift α is introduced:

$$\int_{r_t}^{R} k_r dr = (n + \alpha)\pi, \qquad \text{with} \qquad \frac{c(r_t)}{r_t} = \frac{\omega}{(l(l+1))^{1/2}}. \qquad (4.43)$$

This results in *Duvall law* (Duvall 1982; Nature 300, 242):

$$F(\frac{\omega}{L}) = \int_{r_t}^{R}\left(1 - \frac{L^2c^2}{\omega^2r^2}\right)^{1/2}\frac{dr}{c} = \frac{[n + \alpha(\omega)]\pi}{\omega},$$
$$L = l + \frac{1}{2}. \qquad (4.44)$$

c is a function of r. Fig. 4.1 shows the results of Duvall's law for a frequency where α is taken as 1.5.

4.4.2 More rigorous analysis

Gough generalized the Duvall's law. It is completely given in Deubner & Gough, 1989 of an analysis by Lamb (1909). They assume:

- Cowling's approximation,
- no derivatives of the gravitational acceleration, a quasiplane-parallel approximation, and
- allow thermosdynamic variations, including γ_1.

They define:

$$X = c^2 \rho^{1/2} \nabla \xi \tag{4.45}$$

and retrieve:

$$\frac{d^2X}{dr^2} + \frac{X}{c^2}\left[S_l^2\left(\frac{N^2}{\omega^2} - 1\right) + \omega^2 - \omega_c^2\right] = 0, \tag{4.46}$$

where

$$H = -\left(\frac{d(\ln \rho)}{dr}\right)^{-1} : \qquad \text{density scale height,}$$

$$\omega_c = \frac{c^2}{4H^2}\left(1 - 2\frac{dH}{dr}\right) : \qquad \text{accoustic cut-off frequency.}$$

On transforming eq. (4.46) and taking a factor of ω^2 from the brakets, the equation changes to:

$$\frac{d^2X}{dr^2} = -\frac{X}{c^2\omega^2}\left[S_l^2\left(N^2 - \omega^2\right) + \omega^4 - \omega_c^2\omega^2\right], \tag{4.47}$$

the equation has 2 roots at $(a^2 - \omega^2)$ and $(b^2 - \omega^2)$ which can be retrieved through standard algebra. On introducing the roots in eq. (4.47) the equation can be written to:

$$\frac{d^2X}{dr^2} = -\frac{X}{c^2\omega^2}(\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2), \tag{4.48}$$

where ω_+ is a modified Lamb frequency and ω_- is a modified Bouyancy frequency. Fig. 4.2 shows a frequency plotted vs. the normalized radius of the Sun. The solid line shows the modified Lamb frequency for different degree l, while the broken line the modified Bouyancy frequency for different degrees l shows. Travelling waves have to have a frequency of $\omega \geq \omega_c \simeq 5.3mHz$. The condition for a standing wave can be written as:

$$\omega \int_{r_1}^{r_2} \left[1 - \frac{\omega_c^2}{\omega^2} - \frac{S_l^2}{\omega^2}\left(1 - \frac{N^2}{\omega^2}\right)\right]^{1/2} \frac{dr}{c} \simeq \pi(n - 1/2). \tag{4.49}$$

For the standing wave the factor $\frac{N^2}{\omega^2}$ can be neglected. Then $\frac{\omega_c^2}{\omega^2} \simeq (n - 1/2)$.

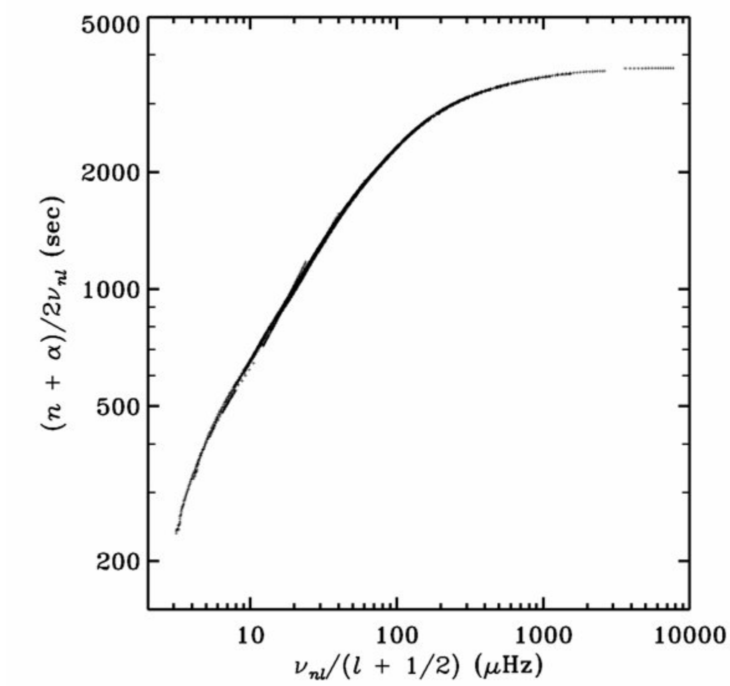

Figure 4.1: Observed Duvall law where α is chosen to 1.5.

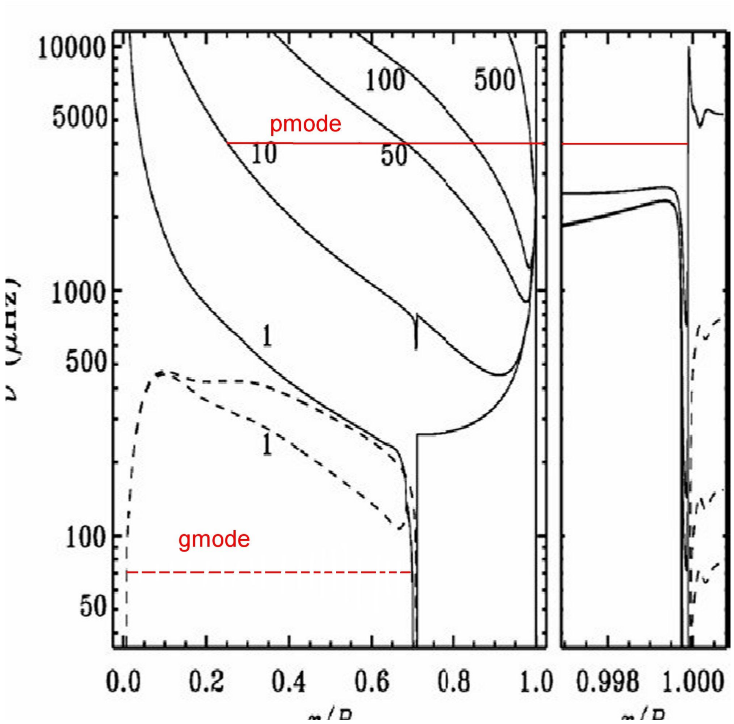

Figure 4.2: The frequency is plotted vs. the normalized Solar radius. The frequencies of a p-mode and the g-mode are inserted for comarison. Additionally the modified Lamb and Bouyancy frequency are plotted for different degrees *l*.

Chapter 5

Inversion of p-mode frequencies

First Abel inversion will be shown for Duvall's law. Afterwards two methods will be described how to solve singularities in the inversion matrix.

5.1 Abel inversion of Duvall's law

Remembering eq. (4.49) with a surface phase shift α Duvall's law can be written to:

$$\int_{r_1}^{r_2} \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{1/2} \frac{dr}{c} = \frac{[n+\alpha]\pi}{\omega}, \tag{5.1}$$

where

$$L = l + \frac{1}{2}. \tag{5.2}$$

On introducing $\varpi = \frac{\omega}{L}$ and $a = \frac{c(r)}{r}$, eq. (5.1) can be transformed to:

$$F(\varpi) = \int_{r_t}^{R} \left(1 - \frac{a^2}{\varpi^2}\right)^{1/2} \frac{dr}{c}. \tag{5.3}$$

F are the observations and $a(r)$ has to be found. After a change of variables, eq. (5.3) is:

$$F(\varpi) = -\int_{a_s}^{\omega} \left(\bar{a}^{-2} - \varpi^{-2}\right)^{1/2} \frac{d(\ln r)}{d\bar{a}} d\bar{a}, \tag{5.4}$$

where:

$$a_s = a(R_\odot); \qquad \frac{c(r_t)}{r_t} = \varpi = a_t; \qquad r = r_t. \tag{5.5}$$

On using (5.5) eq. (5.4) gets to:

$$\frac{dF}{d\varpi} = -\int_{a_s}^{\varpi} \left(\bar{a}^{-2} - \varpi^{-2}\right)^{1/2} \varpi^{-3} \frac{d(\ln r)}{d\bar{a}} d\bar{a}. \tag{5.6}$$

This results in and will be solved in the following:

$$\begin{aligned}
&\int_{a_s}^{a} \left(\varpi^{-2} - \bar{a}^{-2}\right)^{1/2} \frac{dF}{d\varpi} d\varpi \\
&= -\int_{a_s}^{a}\int_{a_s}^{\varpi} \left(\varpi^{-2} - a^{-2}\right)^{-1/2} \left(\bar{a}^{-2} - \varpi^{-2}\right)^{-1/2} \varpi^{-3} \frac{d(\ln r)}{d\bar{a}} d\bar{a} d\varpi \\
&= -\int_{a_s}^{a}\int_{\bar{a}}^{a} \left(\varpi^{-2} - a^{-2}\right)^{-1/2} \left(\bar{a}^{-2} - \varpi^{-2}\right)^{-1/2} \varpi^{-3} \frac{d(\ln r)}{d\bar{a}} d\varpi d\bar{a} \\
&= --\int_{a_s}^{a}\int_{0}^{\frac{\pi}{2}} \frac{d(\ln r)}{d\bar{a}} d\theta d\bar{a} \\
&= -\frac{\pi}{2}\int_{a_s}^{a} \frac{d(\ln r)}{d\bar{a}} d\bar{a} \\
&= -\frac{\pi}{2} \ln\left(\frac{r}{R}\right).
\end{aligned} \tag{5.7}$$

A change of integrations is used to get from line 2 to 3. Fig. 5.1 shows how the

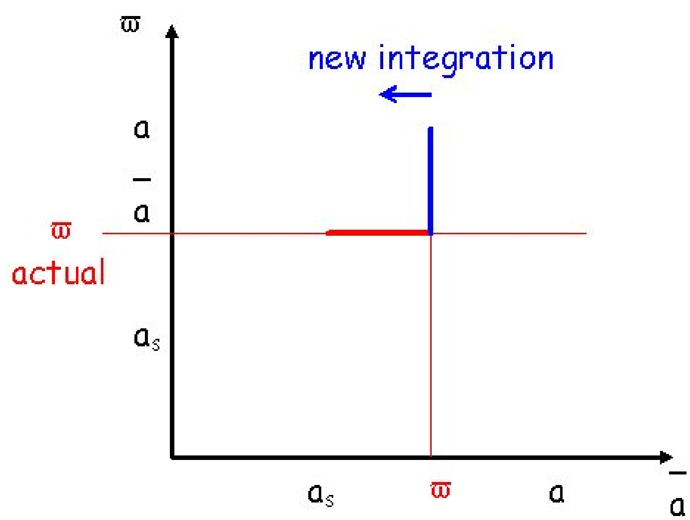

Figure 5.1: Representation for changing the order of integration for getting from line 2 to line 3.

orders of integration are changed. A changing of variables is used $\varpi = a^{-2}\sin^2\theta + \bar{a}^{-2}\cos^2\theta$ for transforming from line 3 to line 4. Finally r can be written to:

$$r = R\exp\left[-\frac{2}{\pi}\int_{a_s}^{a} \left(\varpi^{-2} - \bar{a}^{-2}\right)^{-1/2} \frac{dF}{d\varpi} d\varpi\right]. \tag{5.8}$$

where:

$$a = \frac{c(r)}{r}. \tag{5.9}$$

This gives r as a function of a (fig. 5.2) and hence implicity $a = \frac{c(r)}{r}$ as a function of r and hence c as a function of r.

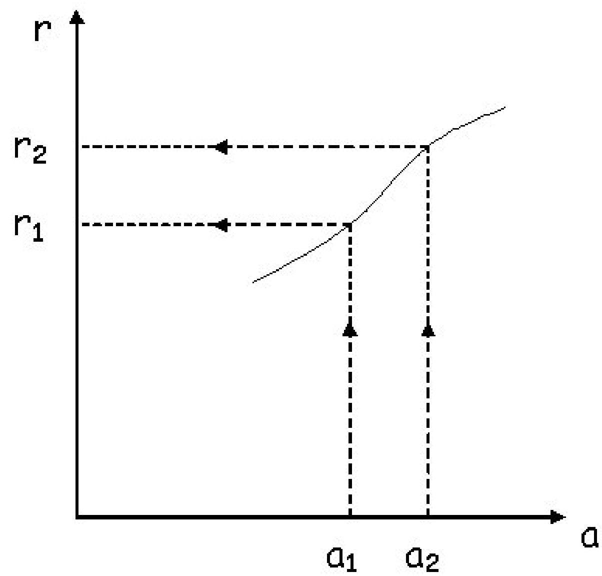

Figure 5.2: Representation for the relation $a = \frac{c(r)}{r}$.

5.2 Regularized Least-Squares (RLS) and Optimally Localized Averages (OLA) inversions

In helioseimology many of the inversion methods which are used are linear. Then the solution is a linear function of the data. First the 1-D rotation law $\Omega(r)$ will be discussed. Then a possibility for regularization of a singular matrix will be shown.

1-D rotation law $\omega(r)$

Rotation raises the degeneracy of a global mode frequencies and introduces a dependence on azimuthal order m. The dependence is particularly simple if a rotation profile $\Omega(r)$ is considered depending only on the rasial coordinate:

$$\omega_{nlm} = \omega_{nl0} + m \int K_{nl}(r)\Omega(r)dr. \tag{5.10}$$

The kernels $K_{nl}(r)$ are different for different modes. $d_{nl} = \frac{(\omega_{nlm} - \omega nl0)}{m}$ are the data. Then

$$d_{nl} = \int K_{nl}(r)\Omega(r)dr + \varepsilon_{nl}, \tag{5.11}$$

where ε_{nl} are noise in the data, each with a standard deviation σ_{nl}. The subscript i is chosen for simplicity in place of "nl".

Least-squares (LS) fitting

The idea of LS is to approximate the unknown function $\omega(r)$ in terms of a chosen set of basis functions $\phi_k(r) : \Omega(r) \approx \bar{\Omega}(r) = \Sigma x_k \phi_k(r)$. The coefficients$x_k$ have to be minimized:

$$\Sigma_i \left(\frac{d_i - \int K_i \bar{\Omega} dr}{\sigma_i} \right)^2 . \tag{5.12}$$

This can be written as a matrix equation:

$$| \mathbf{Ax} - \mathbf{b} |^2 \rightarrow \text{min}. \tag{5.13}$$

The solution of (5.13) is:

$$\mathbf{x} = \left(\mathbf{A}^T \mathbf{A} \right)^{-1} \mathbf{A}^T \mathbf{b}. \tag{5.14}$$

Unfortunately, unless we choose a highly restrictive representation for $\bar{\Omega}$, the matrix $\mathbf{A}$ is usually ill conditioned in helioseismic inversions and so the LS solution $\mathbf{x}$ and hence $\bar{\Omega}$ also are dominated by data noise and thus useless.

In the following two methods for regularization the matrix $\mathbf{A}$ will be shown.

Regularized Least-Squares (RLS) fitting

We can get better-behaved solutions out of LS by adding a ”regularization term” to the minimization, e.g. to minimize:

$$\Sigma \left(\frac{d_i - \int K_i Om\bar{e}gadr}{\sigma_i} \right)^2 = \lambda^2 \int \bar{\Omega}^2 dr, \tag{5.15}$$

$$\Sigma \left(\frac{d_i - \int K_i Om\bar{e}gadr}{\sigma_i} \right)^2 = \lambda^2 \int \left(\frac{d^2 \bar{\Omega}}{dr^2} \right)^2 dr. \tag{5.16}$$

where λ^2 is a trade-off parameter. This can again be written as a matrix equation:

$$| \mathbf{Ax} - \mathbf{b} |^2 + \lambda^2 | \mathbf{Lx} |^2 \rightarrow \text{min}. \tag{5.17}$$

The solution is:

$$\mathbf{x} = \left(\mathbf{A}^T \mathbf{A} + \lambda^2 \mathbf{L}^T \mathbf{L} \right)^{-1} \mathbf{A}^T \mathbf{b}. \tag{5.18}$$

Optimally localized Averages (OLA) method

Since

$$d_i = \int K_i(r) \Omega(r) dr + \varepsilon_i, \qquad i = 1, \ldots, M. \tag{5.19}$$

The idea is to try to find a linear combination of the Kernels for each radial location r_0 that it is localiyed there:

$$\bar{K}(r, r_0) = \Sigma_{i=1}^{M} c_i(r_0) K_i(r). \tag{5.20}$$

If it is successful, then the same linear combination of the data is a localized average of the rotation rate near $r = r_0$:

$$\bar{\Omega}(r_0) \equiv \Sigma c_i d_i = \int (\Sigma c_i K_i) \, \Omega dr + \Sigma c_i \varepsilon_i \quad (5.21)$$

$$= \int \bar{K} \Omega dr + \Sigma c_i\text{''}svarepsilon_i. \quad (5.22)$$

In helioseismology the *Substractive OLA (SOLA)* is mainly used. There the coefficients c_i are so chosen to minimize:

$$\int_0^R \left(\bar{K} - T\right)^2 + \tan\theta \Sigma \sigma_i^2 c_i^2. \quad (5.23)$$

E.g. $T = A \exp\left(\frac{-(r-r_0)^2}{\delta^2}\right)$. This penalizes K for deviating from the target function T. θ and δ are trade-off parameters.