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Exercise for Solar Physics (2008) - Part 1

Chapter 1: Introduction

How is the Sun related to (1) other fields of science and (2) other stars?
How does the Sun affect planets?

Chapter 2: Core and interior

2.1 Solar model

(A) Calculate the Sun's gravitational energy: the total work required to disperse the solar matter over distances $r \gg r_\odot$ ($G = 6.67 \cdot 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$, use the following solar model). Compare this to the total solar irradiance ($1366\,\mathrm{Wm^{-2}}$ at 1 AU) or to the energy in a solar flare (up to $6 \cdot 10^{25}$ J).

$m/m_\odot$	$r/r_\odot$	$T[K]$
0.000	0.000	1.5×10^7
0.125	0.124	1.2×10^7
0.250	0.170	1.0×10^7
0.375	0.210	8.9×10^6
0.500	0.254	7.7×10^6
0.625	0.306	6.6×10^6
0.750	0.367	5.4×10^6
0.875	0.470	4.2×10^6
1.000	1.000	5.8×10^3

(B) Assuming the Sun to be a perfect monatomic gas (pressure $P = \rho RT/\mu$, $R/\mu = 2C_V/3$, gas constant $R = 8.31\,\mathrm{J/(°K\,mol)}$, μ is the mean molecular weight) in hydrostatic equilibrium, calculate the internal energy. (Hint: derive the virial theorem using integration by parts.) Find the mean mass-weighted temperature for such a gas and compare to that from the tabulated model.

2.2 Nuclear reactions

How does the nuclear energy content of the Sun compare to its gravitational and internal energies? (Assume all protons are converted to $\alpha-$particles via ppI. $1\,\mathrm{eV} = 1.602 \cdot 10^{-19}$ J, Avogadro's Number $N_A = 6.0 \cdot 10^{23}$)

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Solutions to Exercise Solar Physics (2008)

Chap.1 Introduction

(1) To other fields of science: astrophysics (stellar evolution, cosmic rays); plasma physics (dynamo, turbulence, waves, fusion); particle physics (atoms, molecules, neutrinos, standard model of elementary particles); gravitation (general relativity); Sun-Earth or -planets relation (space weather, climate).

(2) To other stars: The Sun is a main sequence star (age about 4.5×10^9 yr), evolving into a red giant, planetary nebula, and then a white dwarf.

(3) Relations to planets The Sun provides heat and light at various wavelengths (causing expansion of the atmosphere through heating); The interaction with the Sun involves also the solar wind. For magnetized planets (Earth, Jupiter, Saturn, Uranus, Neptune) the interaction with the Sun or the solar wind causes formation of magnetosphere and various magnetospheric or ionospheric activities such as aurora, radiation belt, storms and substorms. For non-magnetized planets (Venus and Mars) the interaction with the Sun and the solar wind ends up with loss (or escape) of the planetary atmosphere into space (called the ion pickup process). For non-atmospheric planets like Mercury or Earth Moon, the solar wind hits directly on the surface and kicks out surface materials (sputtering). The solar wind is a supersonic (and super-Alfvénic) flow and a standing shock wave (bow shock) forms in front of the planets. Comet tails (ion tails) are also caused by the interaction with the solar wind.

Chap.2 Core and interior

2.1 Solar model

(A) Work for shell mass dm and radius r against attraction of the rest mass m

$$dE = -\int_r^\infty \frac{Gmdm}{r'^2} dr' \quad (1)$$

$$= -\frac{Gmdm}{r} \quad (2)$$

Integrate over m

$$E_G = -\int_0^{m_\odot} \frac{Gmdm}{r} \quad (3)$$

$$= -\frac{Gm_\odot{}^2}{r_\odot} \int_0^{m_\odot} \frac{r_\odot}{r} \frac{m}{m_\odot} \frac{dm}{m_\odot} \quad (4)$$

Approximate by $\int f(x)dx \simeq \sum f(x)\Delta x$,

$$E_G \simeq -\frac{Gm_\odot{}^2}{r_\odot} \times 0.125 \times [0 + 1.008 + \cdots + 1] \quad (5)$$

$$\simeq -6.3 \times 10^{41}[\mathrm{J}], \quad (6)$$

where we used the values $G = 6.67 \times 10^{-11}\,[\mathrm{m^3/kg\,s^2}]$, $m_\odot = 1.99 \times 10^{30}\,[\mathrm{kg}]$, and $r_\odot = 6.96 \times 10^8\,\mathrm{m}$. Hence the Sun's gravitational energy is $|E_G| = 6.3 \times 10^{41}\,[\mathrm{J}]$.

The solar irradiance is $S = 1.336 \times 10^3\,[\mathrm{W/m^2}]$ at 1 AU. Noting that 1 [W] = 1 [J/s] and 1 [AU] $= 1.496 \times 10^{11}$ [m], the total irradiance (integrated over the surface $4\pi r^2$) is $\dot{E}_R = S \times 4\pi r^2 = 3.8 \times 10^{26}$ [J/s]. This is the amount of energy that the Sun provides through radiation per second. With the gravitational energy only, the Sun can provide energy for the period $E_G/\dot{E}_R = 6.3 \times 10^{41}/3.8 \times 10^{26}\,[\mathrm{s}] = 1.7 \times 10^{15}\ [\mathrm{s}] = 5.4 \times 10^7\,[\mathrm{yr}]$. In reality, the Sun's age is about 4.5×10^9 [yr]. So one needs to find an alternative energy source (which is nuclear reaction).

(B) Hydrostatic equilibrium is a force balance between pressure gradient and gravity

$$-\nabla P + \rho\vec{g} = 0 \quad (7)$$

where gravity (or gravitational acceleration) is

$$\vec{g} = -\frac{Gm}{r^2}\vec{e}_r. \quad (8)$$

In the radial direction the hydrostatic balance reads

$$\frac{dP}{dr} = -\frac{Gm\rho^2}{r^2} \quad (9)$$

Using mass-radius relation $dm = 4\pi r^2 \rho dr$, the hydrostatic balance equation gives us the pressure-mass relation

$$dP = -\frac{Gm}{4\pi r^4}dm. \quad (10)$$

Multiply the both sides by sphere volume $V = \frac{4\pi}{3}r^3$,

$$(\mathrm{lhs}) = VdP = d(PV) - PdV \quad (11)$$

$$(\mathrm{rhs}) = -\frac{Gm}{4\pi r^4}\frac{4\pi r^3}{3}dm = -\frac{Gm}{3r}dm. \quad (12)$$

Integrate from the Sun center to the surface

$$\int_{cen}^{sur} (\mathrm{lhs}) = \int_c^s d(PV) - \int_c^s PdV \quad (13)$$

Note that the first term on the right hand side vanishes because $P = 0$ at solar surface and $V = 0$ at center. Use the ideal gas pressure $P = \rho\frac{RT}{\mu} = \rho\frac{2C_V}{3}T$,

$$\int_c^s (\mathrm{lhs}) = -\int_c^s PdV \quad (14)$$

$$= -\frac{2}{3}\int_c^s \rho C_V T dV \quad (15)$$

$$= -\frac{2}{3}\int_0^{m_\odot} C_V T dm \quad (16)$$

$$= -\frac{2}{3}U, \quad (17)$$

here the last integral gives us the internal energy U.

The right hand side of the pressure-mass relation yields one third of the gravitational energy when integrated over m

$$\int_c^s (\text{rhs}) = -\int_c^s \frac{Gm}{3r} dm \tag{18}$$

$$= \frac{1}{3} E_G. \tag{19}$$

Hence we have

$$U + 2E_G = 0, \tag{20}$$

and this is called the *virial theorem*.

The mass-weighted mean temperature is

$$\langle T \rangle = \frac{1}{m_\odot} \int_0^{m_\odot} T dm \tag{21}$$

For constant C_V, the virial theorem yields

$$2C_V \int_0^{m_\odot} T dm = -E_G. \tag{22}$$

The left hand side can also be written with the mean temperature as

$$2C_V \int_0^{m_\odot} T dm = 2C_V m_\odot \langle T \rangle. \tag{23}$$

Hence the virial theorem gives us the mean temperature as

$$\langle T \rangle = -\frac{E_G}{2C_V m_\odot} \tag{24}$$

$$= -\frac{E_G}{m_\odot} \frac{\mu}{3R}, \tag{25}$$

where C_V is replaced by the gas constant $R = 8.31\,[\text{J/K mol}]$. If we take the mean molecular weight $\mu = 0.5\,[\text{g/mol}]$, the mean temperature (estimated from the virial equilibrium) is

$$\langle T \rangle = \frac{|E_G| \times \mu}{m_\odot \times 3R} \tag{26}$$

$$= \frac{6.25 \times 10^{41}\,[\text{J}] \times 0.5 \times 10^{-3}\,[\text{kg/mol}]}{3 \times 8.31\,[\text{J/K mol}] \times 1.99 \times 10^{30}\,[\text{kg}]} \tag{27}$$

$$= 6.30 \times 10^6\,[\text{K}]. \tag{28}$$

On the other hand, the tabulated solar model gives us the mean temperature

$$\langle T \rangle_{\text{tab}} = \int_0^{m_\odot} T \frac{dm}{m_\odot} \simeq 7.83 \times 10^6\,[\text{K}], \tag{29}$$

which is close to the virial temperature. The Sun is therefore (roughly speaking) in a hydrostatic and virial equilibrium.

2.2 Nuclear reactions

The ppI reaction releases the energy about 26.732 [MeV] from 4 protons,

$$4\,(\text{protons}) \rightarrow 1\,(\alpha\text{−particle}) + 26.732\,[\text{MeV}]. \tag{30}$$

In other words, the energy release is $\Delta E_{\rm nuc} = 26.732/4\,[\text{MeV}] = 6.683\,[\text{MeV}] = 6.683 \times 10^6 \times 1.602 \times 10^{-19}\,[\text{J}] = 1.071 \times 10^{-12}\,[\text{J}]$ per proton. The number of protons can be simply estimated as $N = m_\odot/m_p = 1.99 \times 10^{30}\,[\text{kg}]/1.67 \times 10^{-27}\,[\text{kg}] = 1.192 \times 10^{57}$ [particles] on the assumption that the Sun entirely consists of protons. The total energy release from the nuclear reaction is $E_{\rm nuc} = N\Delta E_{\rm nuc} = 1.3 \times 10^{45}\,[\text{J}]$. This is about 2000 times larger than the gravitational energy. When compared to the solar irradiance, the nuclear reaction provides the energy for the period $E_{\rm nuc}/\dot{E}_R = 1.3 \times 10^{45}\,[\text{J}]/3.8 \times 10^{26}\,[\text{J/s}] = 3.4 \times 10^{18}\,[\text{s}] = 1.1 \times 10^{11}\,[\text{yr}]$, which can account for the energy source problem of the Sun. (cf. the Sun's age is $4.5 \times 10^9\,[\text{yr}]$) Compared to the internal energy, $E_{\rm nuc}/U = E_{\rm nuc}/(1/2E_G) \sim 4000$.

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Exercise for Solar Physics (2008) - Part 2

Chapter 3: Oscillations

The f-mode corresponds closely to (deep) ocean waves. In these two questions we will try to get some insight into their behaviour. The first part shows you the mathematics behind this type of solution. It is worth doing, and the answer is at the bottom in case you get stuck. The second part asks you to think about the mathematical solution and the actual Sun.

3.1 Local helioseismology & the f-mode (1)

We begin with the momentum equation

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\nabla P' - \rho' g \hat{\mathbf{z}} \tag{1}$$

where ξ is the displacement caused by the waves, and P' and ρ' are the pressure and density pertubations associated with the waves. The continuity equation is

$$\rho' = -\nabla \cdot (\rho_0 \xi) \tag{2}$$

and the equation of state and energy equation (here adiababitc) can be combined to give

$$P' = c_0^2(\rho' + \xi \cdot \nabla \rho_0) - \xi \cdot \nabla P_0 \tag{3}$$

The subscript 0 denotes the background solar stratification which we will assume varies only in the z direction.

To keep life simple we will only consider a small patch of the solar surface (so we can use a cartesian coordinate system x, y, z). To not lose contact with waves on the surface of the ocean we look for solutions which work for both the Sun and the ocean. We keep the full equations, but look for solutions which *additionally* do not compress the gas, i.e., $\nabla \cdot \xi = 0$. Note carefully that we are using the full equations and only looking for solutions obeying this additional constraint. Such solutions need not exist, and hence we need to remember to check the solutions at the end for consistency.

Using equation 2, show $\nabla \cdot \xi = 0$ implies $\rho' + \xi \cdot \nabla \rho_0 = 0$.

Then assume that the atmosphere is hydrostatic ($\nabla P_0 = -\rho_0 g \hat{\mathbf{z}}$) to obtain (from equation 3) $P' = \xi_z \rho_0 g$.

This and $\rho' = -\xi \cdot \nabla\rho_0$ can be used to eliminate P' and ρ' from the momentum equation.

Remembering that the background atmosphere ρ_0 varies only in the z direction, expand what remains until you have an equation for $\partial^2\xi/\partial t^2$ where none of P', ρ', ρ_0, P_0 and c_0 appear.

Write down the z-component of this equation. Write down the x-component of this equation.

At this stage you should have something like

$$\frac{\partial^2 \xi_z}{\partial t^2} = -g\frac{\partial \xi_z}{\partial z}$$
$$\frac{\partial^2 \xi_x}{\partial t^2} = -g\frac{\partial \xi_z}{\partial x}$$

Pretty.

Find some solutions with $\xi_z = e^{-i\omega t} \times e^{ik_x x} \times e^{k_z z}$, $\xi_x = ie^{-i\omega t} \times e^{ik_x x} \times e^{k_z z}$. You will also need to use $\nabla \cdot \xi = 0$ (which means for our solutions $ik_x\xi_x = -k_z\xi_z$). [At home check the solution satisfies all the equations.]

Find the dispersion relationship $\omega = f(k_x)$.

Bonus question: What are the group and phase speeds (both for the ocean and the Sun)?

[Answer: $k_x = k_z$, and $\omega = \sqrt{gk_x}$, i.e.,

$\xi_z = e^{\pm i\omega t} \times e^{ik_x x} \times e^{k_x z}$,

$\xi_x = ie^{\pm i\omega t} \times e^{ik_x x} \times e^{k_x z}$]

3.2 Local helioseismology & the f-mode (2)

[Starting from the answer to question 1 and noting that the velocity associated with the wave is just $i\omega\xi$.]

Draw some of these solutions as a function of z. Where is most of the kinetic energy? [hint: draw a rough sketch of the density and of the behaviour of the veloicty as a function of height. What are the different possibilities depending on k_x?]

Why are the solutions for the deep ocean only? (What does "deep" mean here?)

How bad are the approximations we used to get these results (when are we safe with our assumptions)?

How do you think convection affects the waves? [Hint: Ocean waves can propagate a long way on a smooth sea.] What about the chromosphere? Roughly sketch how you expect the lifetimes of the modes to depend upon k_z. [Hint: The chromosphere is a very violent place for the modes we are discussing.]

Chapter 4: Rotation

In many models the 11 year solar cycle is thought to be bound up with the solar differential rotation, converting “poloidal” flux to “toroidal” flux.

1) Draw a cut through the sun showing the core, the radiative zone and the convection zone.

2) Draw a magnetic field line passing from the just to the right of the north pole to just to the right of the south pole. Note that the field line doesn’t have time to penetrate the radiative core and hence makes a detour.

3) What is the rotation rate at the pole? How many times does the field-line rotate about the axis per year? (Assume the field line is “frozen” into the plasma.)

4) What is the rotation rate at the equator? How many times does it get wound up here?

5) If there was no differential rotation what would the answer to 3 and 4 be?

6) If there was no differential rotation how would the fieldline look after a year?

7) How does it look with the solar differential rotation?

8) What if a star had more differential rotation?

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Solutions to Exercise Solar Physics (2008)

Chap.3 Oscillations

Part 1. Continuity equation:

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0. \tag{1}$$

Replace the time derivative by $\delta\rho/\delta t$, multiply the equation by δt,

$$\delta\rho + \nabla \cdot (\rho \vec{v} \delta t) = 0. \tag{2}$$

Use displacement $\vec{\xi} = \delta\vec{x} = \vec{v}\delta t$, and we obtain Eq. (2) in the sheet:

$$\delta\rho = -\nabla \cdot (\rho_0 \vec{\xi}). \tag{3}$$

Pressure variation is derived from the definition of the sound speed:

$$c_s^2 = \frac{P}{\rho}, \tag{4}$$

where we set the polytropic index (or ratio of specific heat) $\gamma = 1$. The fluctuation of the pressure is

$$\delta P = c_s^2 \delta\rho \text{ (adiabatic)}, \tag{5}$$

and we replace $\delta\rho \to \delta\rho + \vec{\xi} \cdot \nabla\rho_0$, where the first term denotes the fluctuation (oscillation or wave field) of density and the second term the change of the background. We also replace the pressure as $\delta P \to \delta P + \vec{\xi} \cdot \nabla P_0$, and here again the first term is the fluctuation of the pressure and the second term the change of the background. The pressure variation is written in the form

$$\delta P + \vec{\xi} \cdot \nabla P_0 = c_s^2 (\delta\rho + \vec{\xi} \cdot \nabla\rho_0), \tag{6}$$

which gives Eq. (3) in the sheet:

$$\delta P = c_s^2 (\delta\rho + \vec{\xi} \cdot \nabla\rho_0) - \vec{\xi} \cdot \nabla P_0. \tag{7}$$

Incompressibility means $\nabla \cdot \vec{v} = 0$. Multiply by δt and we obtain $\nabla \cdot \vec{\xi} = 0$. The continuity equation becomes

$$\delta\rho + \vec{\xi} \cdot \nabla\rho_0 = 0. \tag{8}$$

Hydrostatic equilibrium in the z (vertical) direction is

$$-\nabla P_0 - \rho_0 g \vec{e}_z = 0. \tag{9}$$

In the pressure variation equation the first term with the bracket vanishes because of the continuity equation under incompressibility $\delta\rho + \vec{\xi}\cdot\nabla\rho_0 = 0$. The pressure variation is hence

$$\begin{aligned}\delta P &= -\vec{\xi}\cdot\nabla P_0 &(10)\\ &= \xi_z \rho g, &(11)\end{aligned}$$

where the ∇-part is replaced by the density fluctuation using the hydrostatic balance.

Now we use the pressure variation

$$\delta P = \xi_z \rho_0 g \tag{12}$$

and the density variation

$$\delta\rho = -\vec{\xi}\cdot\nabla\rho_0 \tag{13}$$

in the momentum equation (for waves),

$$\begin{aligned}\rho_0 \partial_t^2 \vec{\xi} &= -\nabla(\delta P) - \delta\rho g \vec{e}_z &(14)\\ &= -\nabla(\xi_z \rho_0(z) g) + \vec{\xi}\cdot\nabla\rho_0(z) g &(15)\\ &= -\nabla(\xi_z \rho_0(z) g) + \xi_z \partial_z \rho_0(z) g. &(16)\end{aligned}$$

In the z component the right hand side of the momentum equation is written as

$$\begin{aligned}(\text{rhs}) &= -\partial_z(\xi_z \rho_0(z) g) + \xi_z \partial_z(\rho_0(z) g) &(17)\\ &= -\rho_0 g \partial_z \xi_z. &(18)\end{aligned}$$

Here we take $g = \text{const}$. The density ρ_0 in the momentum equation is canceled out and we obtain the momentum equation in the form

$$\partial_t^2 \xi_z = -g\partial_z \xi_z. \tag{19}$$

For the x component,

$$\begin{aligned}(\text{rhs}) &= -(\partial_x \xi_z)\rho_0(z) g &(20)\\ (\text{lhs}) &= \rho_0(z)\partial_t^2 \xi_x, &(21)\end{aligned}$$

hence

$$\partial_t^2 \xi_x = -g\partial_x \xi_z. \tag{22}$$

Pretty.

We use the ansatz

$$\begin{aligned}\xi_z &= \exp[-i(\omega t - k_x x) + k_z z] &(23)\\ \xi_x &= i\exp[-i(\omega t - k_x x) + k_z z]. &(24)\\ & &(25)\end{aligned}$$

This means that we have a plane wave propagation in the x direction with the amplitude unity at base ($z = 0$). In the z direction the wave amplitude grows exponentially. The displacement in the x direction (ξ_x) has a phase shift by $\pi/2$ from that of ξ_z, such that the oscillation of fluid element forms a circular motion in the xz plane.

When we use the ansatz, we obtain the dispersion relation as a solution of the equations, and that is

$$\omega^2 = gk_x = gk_z. \tag{26}$$

The wave propagates solely in the x direction, and the phase speed is

$$v_{ph} = \frac{\omega k_x}{|k^2|} = \frac{\omega}{k_x} = \frac{g}{\omega}. \tag{27}$$

The group speed is

$$v_{gr} = \frac{\omega}{k_x} = \frac{g}{2\omega} = \frac{1}{2} v_{ph}. \tag{28}$$

Part 2. Hydrostatic equilibrium is given as

$$-\nabla P_0 - \rho_0 \vec{g} = 0. \tag{29}$$

We use the ideal gas for the pressure,

$$P_0 = n_0 kT = \rho_0 \frac{kT}{m}, \tag{30}$$

where m is the mean molecular weight. Combining the two equations, we obtain

$$\nabla \rho_0 = -\frac{m\vec{g}}{kT} \rho_0, \tag{31}$$

where we assumed an isothermal gas ($T = \text{const}$). The gravity is in the z direction only, and

$$\frac{d\rho_0}{dz} = -\frac{mg}{kT} \rho_0, \tag{32}$$

which can easily be solved and the solution is an exponential decay of the density:

$$\rho_0(z) = \rho_0(0) e^{-z/H}, \tag{33}$$

where H is called the scale height

$$H = \frac{mg}{kT}. \tag{34}$$

The velocity of the medium associated with the wave oscillation is

$$\vec{v} = i\omega \vec{\xi}. \tag{35}$$

We use the dispersion relation

$$\omega^2 = gk_x, \tag{36}$$

which gives the squared velocity

$$\begin{aligned} |\vec{v}|^2 &= \omega^2 |\vec{\xi}|^2 & (37) \\ &= gk_x (|\xi_x|^2 + |\xi_z|^2) & (38) \\ &= 2gk_x e^{2k_z z}. & (39) \end{aligned}$$

The kinetic energy of the oscillation is hence

$$\begin{aligned}\frac{1}{2}\rho_0 v^2 &= \frac{1}{2}(\rho_0(0)e^{-z/H})(2gk_x e^{2k_z z}) && (40)\\ &= \rho_0(0)e^{(-1/H+2k_z)z}. && (41)\end{aligned}$$

If $(-1/H + 2k_z) > 0$, the kinetic energy grows in the z direction. If $(-1/H + 2k_z) < 0$, the energy decays in the z direction. (See discussion in Airy wave theory in fluid dynamics.)

The wave mode is called "deep" because the scale height H is large enough for wave energy to grow vertically, $(-1/H + 2k_z) > 0$.

In deriving the deep ocean wave mode we assumed that the gravitational acceleration is constant, but in reality the gravity should be a function of the radius, $g = \text{const} \to g = -Gm/r^2$. Also we have linearized the equations, neglected all the nonlinear terms (for example the advection term in the momentum equation). This is valid when the wave amplitude is small compared to the background field. We also used the adiabatic change of gas (eq. of state), but in reality we have convection which violates the adiabatic change.

The waves interact with convection motion when the wavelength is close to the convection cell size. If the wavelength is smaller than the cell size the wave can no longer propagate, but becomes distorted, convected, or broken by the cells.

lifetime?

Chap.4 Rotation

See winding of magnetic field (Ω-effect, on the next page). The winding develops until turbulence or convection starts to twist the toroidal magnetic field into the poloidal field (α-effect). The combination of these two effects makes the Sun's magnetic field oscillatory at 11 year cycle. Theoretically the magnetic field may grow until all the kinetic energy is converted to the magnetic field energy (magnetic braking).

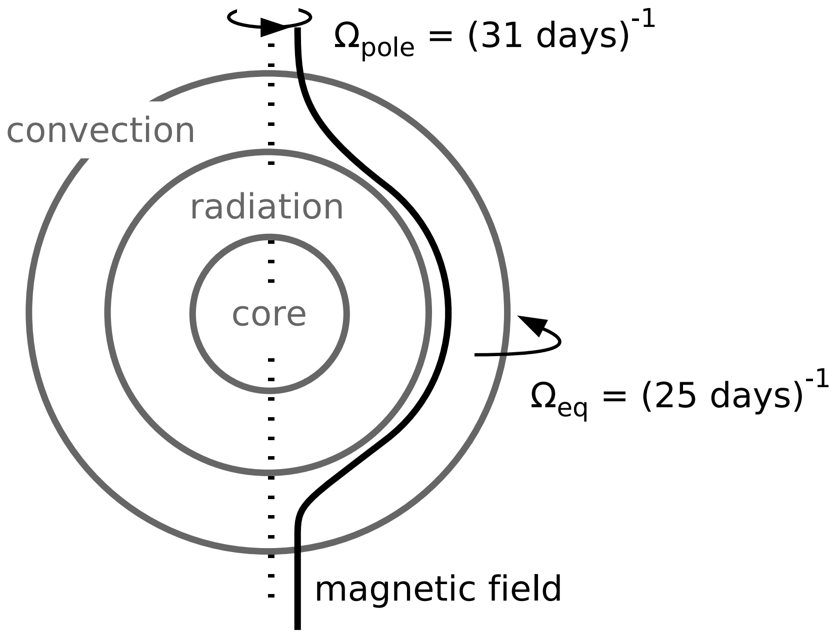

Figure 1: Before winding of magnetic field.

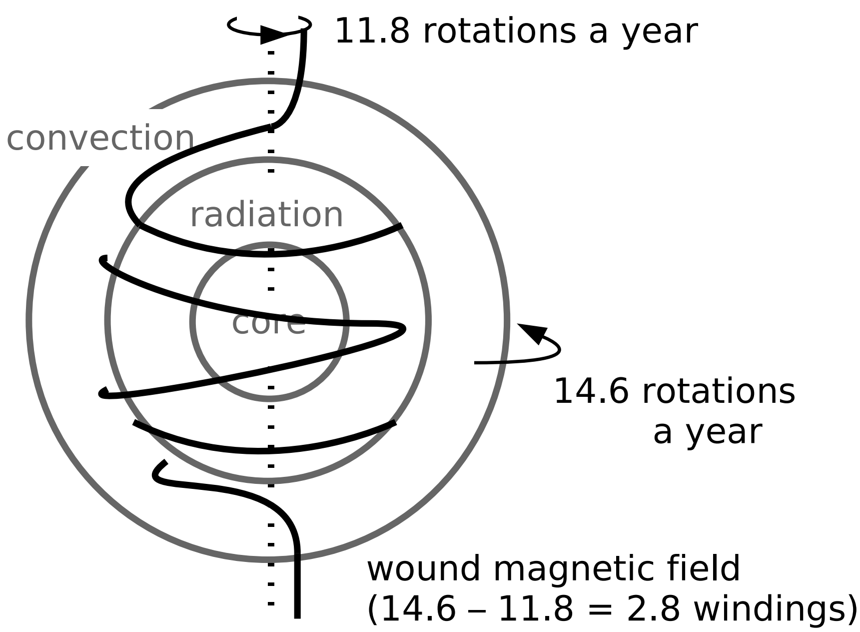

Figure 2: Magnetic field one year later.

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Exercise for Solar Physics (2008) - Part 3

Chapter 5: Radiation and spectrum

5.1 Optical depth

Give a physical interpretation of optical depth.

5.2 Limb darkening

(A) What is limb darkening?

(B) The limb darkening can be approximated by $I(\theta) \sim \cos\theta$ to the first order. Here $I(\theta)$ denotes the intensity at the angle to the local vertical direction θ (for the optical depth $\tau = 0$).

1. Derive the surface intensity at $\tau = 0$ in the form

$$I(0,\theta) = \int_0^\infty S(\tau) e^{-u} du$$

from the radiative transfer equation

$$\cos\theta \frac{dI(\tau,\theta)}{d\tau} = I(\tau,\theta) - S(\tau)$$

by multiplying by an integrating factor $\exp(-\tau/\cos\theta) = \exp(-u)$. $S(\tau)$ is called the source function.

2. Assume a linear dependence for the source function

$$S(\tau) = a + b\tau$$

and obtain the cosine dependence for the surface intensity

$$I(0,\theta) = a + b\cos\theta,$$

where a and b are constants.

Chapter 6: Convection

6.1 Onset of convection

The onset of the convection can be derived from the argument of entropy. A fluid element is convectively unstable if its entropy decreases in a certain direction (outward in the radial direction in the solar physics context), viz.,

$$ds < 0 \quad \text{(radially outward)},$$

where the entropy is defined as

$$s = \frac{P}{\rho^\gamma}.$$

Here P and ρ denote the pressure and the mass density of the fluid element, respectively, and γ the polytropic index (the ratio of specific heat). Note that we do not give the equation of state yet.

1. Derive the condition for convection in the form

$$\frac{d(\log P)}{dr} - \gamma \frac{d(\log \rho)}{dr} < 0 \quad \text{(radially outward)},$$

where dr denotes the line element in the radial direction.

2. Derive the condition for convection, on the assumption of the equation of state for ideal gas, in the form

$$\frac{d(\log T)}{d(\log P)} < \frac{\gamma - 1}{\gamma} \quad \text{(radially outward)},$$

where T denotes the temperature. In the solar physics the left hand side represents the temperature gradient given by radiative energy transport (∇_{rad}), and the right hand side the adiabatic gradient (∇_{ad}). In the radially *inward* direction (often used in the solar physics) the condition reads as $\nabla_{\mathrm{rad}} > \nabla_{\mathrm{ad}}$.

6.2 Granulation model

Consider a mass flow $\rho\vec{v}$ that horizontally varies like $\sin kx$, and has a vertical scale height H. Show that the ratio of horizontal to vertical velocity components scales as

$$\frac{v_h}{v_r} \simeq \frac{1}{kH}.$$

For given H, therefore, the ratio v_h/v_r increases with increasing cell size.

Chapter 7: Atmosphere

7.1 Corona temperature

In place of the strong Calcium (Ca^+) H and K lines (wavelengths $\lambda \simeq 400\,\mathrm{nm}$) rather shallow and broad dips (width $\Delta\lambda \simeq 20\,\mathrm{nm}$) can be noticed in the spectrum of K corona. Assume that the dips are in fact the Doppler broadening of the line spectrum and derive a temperature of particles. Note that Thomson scattering (photon-electron scattering) is one of the major processes in the K corona. In this way Grotrian (1931) first concluded that the corona might be hot.

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Solutions to Exercise Solar Physics (2008)

Chap.5 Radiation

Optical depth

The optical depth τ is defined in a differential form as

$$d\tau_\nu = -\kappa_\nu dz, \tag{1}$$

where κ is the absorption coefficient in units of [m^{-1}] and z is the length in the line-of-sight direction in units of [m]. The minus sign on the rhs means that we look in the direction toward the cloud. The absorption coefficient is a measure of the inverse mean free path of photons in the cloud, so the optical depth compares the mean free path with the length dz at frequency ν or wavelength λ. The optical depth is dimensionless and can be interpreted as the number of mean free paths through the length dz in the cloud.

We can also argue with the radiative transfer equation. Assuming no emission in the cloud and fixing the direction $\theta = 180°$, the equation has a simple form

$$-\frac{dI}{d\tau} = I, \tag{2}$$

which can be easily solved to

$$I = I_0 e^{-\tau}. \tag{3}$$

Therefore at $\tau = 1$ the intensity I (or the number of photons) is diminished by $1/e$ (about 37%).

Limb darkening

(A) Around the disk center one sees the atmosphere almost vertically. The atmospheric layer becomes hotter as one goes deep into the atmosphere, and therefore at the layer $\tau = 1$ one sees a bright surface (Stefan-Boltzmann). Near the limb, on the other hand, one sees the atmosphere almost in the horizontal direction and the layer of $\tau = 1$ is higher than that of the disk center. At higher layers the temperature is lower, therefore the surface looks darker. The decrease of intensity from the disk center to the limb scales with $\cos\theta$, where θ is the angle from the vertical (or radial) direction (see Figure).

(B) Radiative transfer equation is

$$\cos\theta \frac{dI}{d\tau} = I - S. \tag{4}$$

Multiply the equation by factor $\exp[-\tau/\cos\theta] = \exp[-u]$ (where $u = \tau/\cos\theta$ and $\theta = \text{const}$),

$$\left[\frac{dI}{du} - I\right] e^{-u} = -Se^{-u} \tag{5}$$

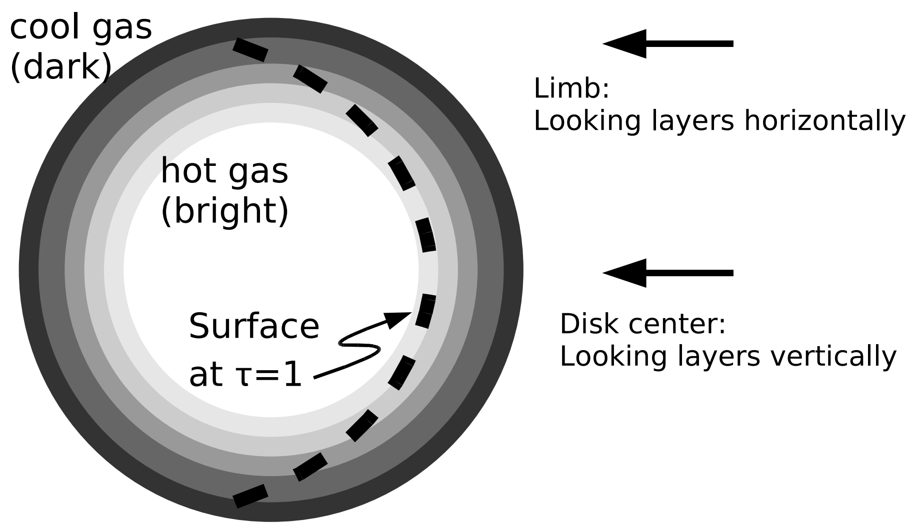

Figure 1: Limb darkening.

The left hand side can become simpler:

$$\frac{d}{du}\left(Ie^{-u}\right) = -Se^{-u}. \tag{6}$$

Integrate from $u = 0$ to ∞,

$$(\mathrm{lhs}) = \left[Ie^{-u}\right]_0^\infty = -I_{\tau=0} = -I_0 \tag{7}$$

$$(\mathrm{rhs}) = -\int_0^\infty Se^{-u}du, \tag{8}$$

here I_0 is the intensity at $\tau = 0$.

Hence we have

$$I_0 = \int_0^\infty Se^{-u}du. \tag{9}$$

Now assume that the source function is linear to the optical depth

$$S = a + b\tau, \tag{10}$$

which means that the source function decreases linearly to the height z,

$$S = a - b\kappa z. \tag{11}$$

The intensity can be obtained by integration

$$I_0 = \int_0^\infty ae^{-u}du + \int_0^\infty b\tau e^{-u}du \tag{12}$$

The first and the second term yields, respectively,

$$(\text{1st}) = a \int_0^\infty e^{-u} du = a \tag{13}$$

$$(\text{2nd}) = b \cos\theta \int_0^\infty u e^{-u} du = b\cos\theta, \tag{14}$$

here we used the formula

$$\int_0^\infty u^n e^{-u} du = n!. \tag{15}$$

The intensity therefore scales to the cosine function,

$$I_0 = a + b\cos\theta. \tag{16}$$

Chap.6 Convection

Onset of convection

We define the entropy as

$$s = \frac{P}{\rho^\gamma}. \tag{17}$$

Entropy actually measures the logarithm of this quantity, $\log(P/\rho^\gamma)$, but we do not use the log-function in the following calculation for simplicity. The derivative of s with respect to the radial distance r is

$$\begin{aligned} \frac{ds}{dr} &= \frac{dP}{dr}\frac{1}{\rho^\gamma} + P(-\gamma)\frac{1}{\rho^{\gamma+1}}\frac{d\rho}{dr} & (18)\\ &= \frac{1}{\rho^\gamma}\left(\frac{dP}{dr} - \frac{\gamma P}{\rho}\frac{d\rho}{dr}\right) & (19)\\ &= \frac{P}{\rho^\gamma}\left(\frac{1}{P}\frac{dP}{dr} - \gamma\frac{1}{\rho}\frac{d\rho}{dr}\right) & (20)\\ &= \frac{P}{\rho^\gamma}\left(\frac{d(\log P)}{dr} - \gamma\frac{d(\log\rho)}{dr}\right). & (21)\end{aligned}$$

Hence the condition $ds < 0$ means

$$\frac{d(\log P)}{dr} - \gamma\frac{d(\log\rho)}{dr} < 0 \text{ (radially outward).} \tag{22}$$

Now use the ideal gas pressure

$$\begin{aligned} P &= nkT & (23)\\ &= \rho\frac{kT}{\mu}, & (24)\end{aligned}$$

where μ is the mean molecular mass. The mass density is therefore

$$\rho = \frac{\mu P}{kT}. \tag{25}$$

The log-derivative is

$$\frac{d}{dr}(\log \rho) = \frac{d}{dr}\left[\log\left(\frac{\mu P}{kT}\right)\right] \tag{26}$$

$$= \frac{d}{dr}(\log P - \log T). \tag{27}$$

Substitution into the convection condition:

$$\frac{d}{dr}(\log P) - \gamma \frac{d}{dr}(\log \rho) < 0. \tag{28}$$

The left hand side can be written as

$$(\text{lhs}) = \frac{d}{dr}(\log P) - \gamma \frac{d}{dr}(\log P - \log T) \tag{29}$$

$$= (1-\gamma)\frac{d(\log P)}{dr} + \gamma \frac{d(\log T)}{dr}. \tag{30}$$

Drop dr, and we obtain

$$(1-\gamma)d(\log P) + \gamma d(\log T) < 0. \tag{31}$$

or after an arrangement

$$\frac{d(\log T)}{d(\log P)} < \frac{\gamma - 1}{\gamma} \text{ (radially outward)}. \tag{32}$$

In solar physics (and stellar physics, too) the direction is often taken as radially inward, so replacing as $r \to -r$, and we obtain

$$\frac{d(\log T)}{d(\log P)} > \frac{\gamma - 1}{\gamma} \text{ (radially inward)}. \tag{33}$$

The left hand side represents the temperature gradient associated with radiative energy transport, ∇_{rad}, whereas the right hand side is the adiabatic gradient, ∇_{ad}. Hence the condition for onset of convection is

$$\nabla_{\text{rad}} > \nabla_{\text{ad}} \text{ (radially inward)}. \tag{34}$$

Granulation model

Continuity equation under stationary condition is

$$\nabla \cdot (\rho \vec{v}) = 0. \tag{35}$$

The horizontal flow is periodic, $v_h \sim \sin(kx)$, and the left hand side of the continuity equation is for the horizontal scale

$$|\nabla \cdot (\rho \vec{v})| \sim k \rho v_h. \tag{36}$$

For the vertical scale we have

$$|\nabla \cdot (\rho \vec{v})| \sim \frac{\rho v_r}{H}. \tag{37}$$

Balancing the two mass flux scales gives

$$\frac{\rho v_r}{H} \sim k \rho v_h, \tag{38}$$

which results in

$$\frac{v_h}{v_r} \sim \frac{1}{kH}. \tag{39}$$

Chap.7 Atmosphere

Doppler shift relation

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} \tag{40}$$

and thermal speed

$$v_{th}^2 = \frac{2kT}{m} \tag{41}$$

give the relation

$$\left(\frac{\Delta\lambda}{\lambda}\right)^2 = \frac{2kT}{mc^2}, \tag{42}$$

which can be arranged to

$$kT = \frac{1}{2}\left(\frac{\Delta\lambda}{\lambda}\right)^2 mc^2. \tag{43}$$

Use the electron mass $mc^2 = 511\,[\mathrm{keV}]$, line width $\Delta\lambda = 20\,[\mathrm{nm}]$, and wavelength $\lambda = 400\,[\mathrm{nm}]$. We obtain $kT = 6.39\,[\mathrm{eV}] = 7.4 \times 10^6\,[\mathrm{K}]$.

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Exercise for Solar Physics (2008) - Part 4

Chapter 8: Magnetic fields and atmospheric dynamics

8.1 Magnetic pressure

Where does the expression for magnetic pressure come from? Can you see why it called magnetic pressure? Another term results from this as well, what is its function?

You will need here the MHD momentum equation ($\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}$), Ampère's law ($\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$) and a certain triple vector identity ($\mathbf{A} \times (\nabla \times \mathbf{A}) = (\nabla \mathbf{A}) \cdot \mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{A} = \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{A}$).

8.2 Flux tubes and the canopy

In an isothermal system the external pressure, p_e, and the pressure inside a small flux tube, p_i, vary vertically with the same scale height, H: $p_e = p_{e0} e^{-z/H}$ and ditto for p_i. If we assume a pressure balance exists between the tube and its surroundings ($p_{tot,e} = p_{tot,i}$), how does the magnetic field strength vary with height? How about the radius of the tube (note that the magnetic flux in the tube must be conserved)?

If we have a collection of these tubes with a density of n tubes per surface area, at what height will the tubes merge, i.e., above which height is the entire volume filled with magnetic field? How does it depend on B_0 and the spatial average magnetic field strength on the surface?

On the next page you can find a figure of a semi-empirical plane-parallel solar atmospheric model which mimics the quiet Sun (VALC, Vernezza et al., *Astrophys. J. Suppl.*, *45*, 635, 1981). From this you can get a rough estimate for what the scale height is. Now compute the merging height if B_0 is 1.5 KG assuming that the average magnetic field is 4 G (quiet areas) or 200 G (plage).

8.3 Plasma-β

Plasma-β measures the ratio of the gas and magnetic pressures. Where in the VALC model is the plasma-β equal unity if the magnetic field has a constant strength of a) 1000 G, b)100 G, c)10 G? How do you think the $\beta = 1$ layer looks like in the real Sun?

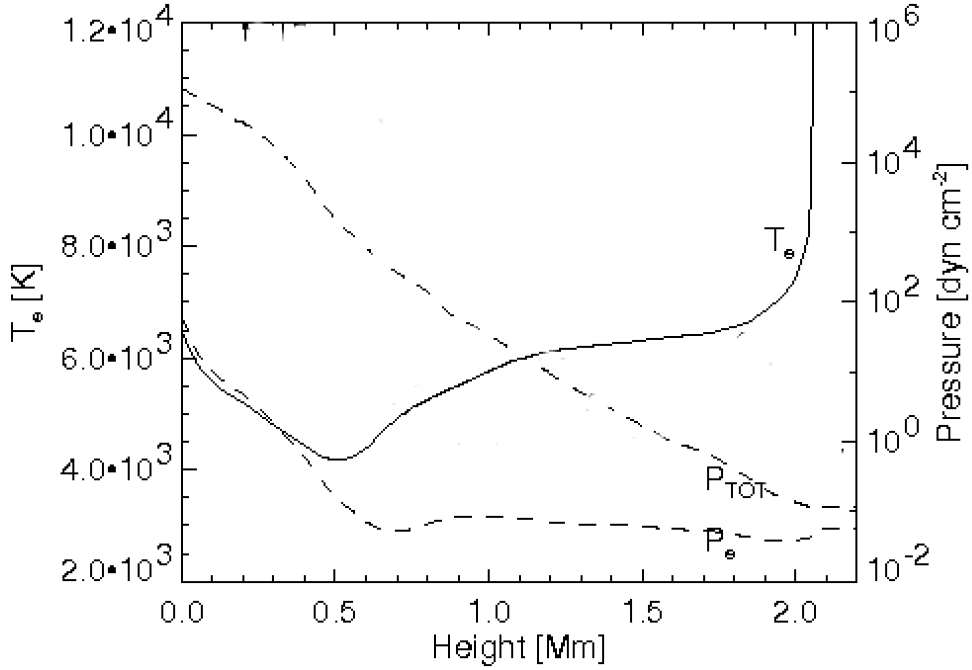

Figure 1: VALC atmospheric model (Vernazza et al., 1981).

8.4 Dynamical time scales

Making realistic simulations of the solar atmosphere is computationally very expensive, especially in the chromosphere, so simplifications and short cuts are needed. One such short cut is assuming statistical equilibrium for the level populations, i.e,. the populations adjust immediately to changing conditions. Is this a valid approximation? (Think in terms of dynamic timescales, e.g., timescale for hydrogen ionization/recombination increases from 1 s in the photosphere to 10^5 s in the mid-chromosphere where it starts to decrease again becoming 10^2 s at the base of the transition region.) Why is hydrogen sometimes referred to as a thermostat?

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Solutions to Exercise Solar Physics (2008)

Chap.8 Magnetic fields and atmospheric duynamics

Magnetic pressure

The Lorenz force $\vec{j} \times \vec{B}$ gives two terms when coupled to the Ampère's law (neglecting the displacement current)

$$\vec{j} \times \vec{B} = \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{B} \tag{1}$$

$$= \frac{1}{\mu_0}(\vec{B} \cdot \nabla)\vec{B} - \nabla \left(\frac{B^2}{2\mu_0} \right). \tag{2}$$

Here the first term is called the magnetic tension (the force acting at curved magnetic field, trying to straighten the field), and the second term is called the magnetic pressure as it resembles the form of the pressure gradient.

8.2 Flux tubes and the canopy

Total pressure is the gas and the magnetic pressure, $P_{\mathrm{tot}} = P_{\mathrm{gas}} + P_{\mathrm{mag}}$. We treat the total pressure constant at each height z. If we apply the constant total pressure inside and outside the flux tube at base ($z = 0$), we have

$$\frac{B_0^2}{2\mu_0} + P_{i0} = P_{e0}, \tag{3}$$

where the left hand side is the total pressure inside the tube and the right hand side outside the tube. The gas pressure itself, on the other hand, is in a hydrostatic equilibrium (to the gravity) and hence it decays exponentially to the scale height H. At height z the total pressure balance reads

$$\frac{B^2(z)}{2\mu_0} + P_{i0}e^{-z/H} = P_{e0}e^{-z/H}. \tag{4}$$

Using the above two equations, we obtain

$$\frac{B(z)}{B_0} = e^{-z/2H}. \tag{5}$$

The magnetic field decreases also with height (but more slowly because the effective scale height is doubled).

Conservation of the magnetic field flux is

$$\Phi = \pi r^2 B = \text{const}, \tag{6}$$

where r denotes the radius of the tube. We compare the flux at different heights,

$$\pi r_0^2 B_0 = \pi r^2 B, \tag{7}$$

where the lhs is at base ($z = 0$) and the rhs is at height z. This gives a scaling of the tube radius as a function of the height

$$\frac{r}{r_0} = e^{+z/4H}, \tag{8}$$

that is, the radius becomes larger with height.

The height of merging is estimated as follows. The average separation of the tubes (from center to center) is about $1/\sqrt{n}$, where n is the density of the tubes. They will merge (contact each other) when the doubled radius (or diameter) reaches the separation distance,

$$r = \frac{1}{2\sqrt{n}}. \tag{9}$$

If we use now the scaling of the tube radius, $r/r_0 = e^{+z/4H}$, we obtain

$$r_0 e^{z/4H} = \frac{1}{2\sqrt{n}}, \tag{10}$$

or when squared,

$$r_0^2 e^{z/2H} = \frac{1}{4n}. \tag{11}$$

Therefore the merging height is

$$z = -2H \log(4nr_0^2), \tag{12}$$

or in terms of B,

$$z = -2H \log\left(\frac{4\langle B\rangle}{\pi B_0^2}\right), \tag{13}$$

where we used the averaged magnetic field

$$\langle B\rangle = n\pi r_0^2 B_0. \tag{14}$$

If we read the scale height H from the plot of P_{tot}, $H \simeq 100$ [km]. Note that at $z = H$ the pressure decreases by $1/e \simeq 0.37$ from the base. For $B_0 = 1500$ [G] and $\langle B\rangle = 4$ [G] (quiet areas) we obtain $z \simeq 1100$ [km]. For $\langle B\rangle = 200$ [G] we obtain $z \simeq 350$ [km]. In both cases the merging height is larger than the scale height H.

8.3 Plasma Beta

The plasma parameter β measures the ratio of the gas to the magnetic pressure,

$$\beta = \frac{P_{\text{gas}}}{P_{\text{mag}}}. \tag{15}$$

At $\beta = 1$ the total pressure can be written only with the magnetic field strength,

$$\begin{aligned} P_{\rm tot} &= 2P_{\rm mag} && (16)\\ &= \frac{B^2}{\mu_0} && (17)\\ &= \frac{B^2}{4\pi \times 10^7}\ [\text{Pa}] && (18)\\ &= \frac{B^2}{4\pi \times 10^7} \times 10\ [\text{dyn/cm}^2], && (19) \end{aligned}$$

where B is in units of T (Tesla). From the plot of $P_{\rm tot}$ we obtain the heights of $\beta = 1$ about 100 [km], 500 [km], and 1200 [km] for 1000 [G], 100 [G], and 10 [G], respectively.

While the surface with the constant total pressure may be smooth, the surface of "(gas pressure) = (magnetic pressure)" is warped and distorted very much in the real Sun (e.g. flux tubes, sunspots). The surface changes not only spatially but also temporally.

8.4 Dynamical time scale

Dynamical time scale in the chromosphere is typically 200-400 [s] (3-minute oscillations in internetwork and 5-7 minutes in network), while the time scale of hydrogen ionization and recombination is of the order 10^5 [s]. The system is therefore constantly evolving, trying to catch up with the dynamics. This results in hydrogen fluctuations that are smaller than those derived from the statistical equilibrium, i.e., the dynamics vary more than the populations. For hydrogen the ionization potential is high and relaxation time to an equilibrium is long. The concept of thermostat applies to the temperature plateau, where energy is used up to ionize hydrogen, which in turn leads to a high specific heat and also releases electrons that can through collisions excite other elements and thus the energy is radiated away instead of a temperture rise. The plateau ends when hydrogen is fully ionized and the transition region begins. On the other hand, the effect during shocks has an opposite sense to the thermostat: because of the long ionization time scales, hydrogen does not have enough time to be ionized during the shock compression phase and energy is used to increase the temperature, i.e., the increase of temperature over shock fronts is sharper when statistical equilibrium is *not* assumed.