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Information on astronomical objects

- Electromagnetic radiation

- Cosmic radiation ($\odot$, Supernovae)

  90 % protons
  10 % α - particles
  Heavy nuclei
  energies : $10^8 \ldots 10^{20}$ eV
  (laboratory : 10^{12} eV)

  diffuse radiation

- Neutrinos ($\odot$, Supernovae)

  Detectors : Chlorine
  Gallium
  Water

- Gravitational radiation

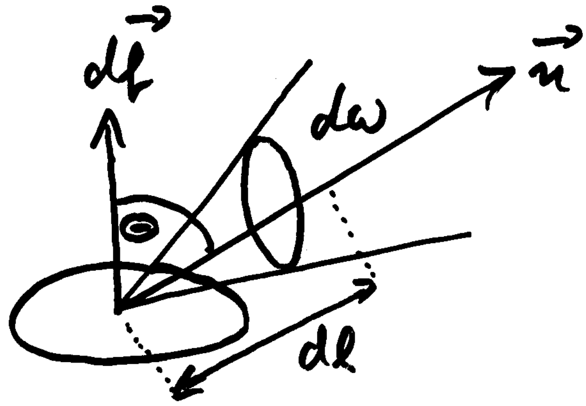
  indirect : binary pulsar
  direct : ?
  (Weber 1960 ?)

Description of radiation field

Intensity I_ν

$$I_\nu = \frac{dE_\nu}{dt\, d\nu\, d\omega\, (\vec{n}\, d\vec{f})}$$

Energy / time / frequency / solid angle / area $\parallel \vec{n}$



$$\vec{n}\, d\vec{f} = \cos\Theta\, |d\vec{f}|$$

$$d\omega = \sin\Theta\, d\Theta\, d\varphi \qquad \varphi \in (0, 2\pi)$$

$$\Theta \in (0, \pi)$$

Angular dependence replaced by description by moments:

$$\int I_\nu \cos^n\Theta\, d\omega$$

0th order moment

$$J_\nu = \frac{1}{4\pi} \int I_\nu \, d\omega \quad \text{mean intensity}$$

physical meaning: consider energy density

$$u_\nu = \frac{dE_\nu}{d\nu \, \vec{n} \, d\vec{f} \, dl} = \frac{dE_\nu}{d\nu \, \vec{n} \, d\vec{f} \, c \, dt}$$

$$= \frac{1}{c} \int I_\nu \, d\omega = \frac{4\pi}{c} J_\nu$$

$$u_\nu = \frac{4\pi}{c} J_\nu$$

1st order moment

$$F_\nu = \left[\frac{1}{\pi}\right] \int I_\nu \cos\Theta \, d\omega \;\hat{=}\; \frac{dE_\nu}{d\nu \, dt \, |d\vec{f}|}$$

radiation flux

Splitting into two parts:

$$F_\nu = \int_0^\pi \int_0^{2\pi} d\Theta \, d\varphi \; I_\nu(\Theta, \varphi) \cos\Theta \sin\Theta$$

$$= \underbrace{\int_0^{\pi/2} d\Theta \ldots}_{F_\nu^+} + \underbrace{\int_{\pi/2}^{\pi} d\Theta}_{-F_\nu^-}$$

2nd order moment

$$K_\nu = \frac{1}{4\pi} \int I_\nu \cos^2\Theta \, d\omega$$

physical meaning: consider pressure

pressure ~ photon momentum $\| d\vec{f}$ / time / area

~ $\frac{1}{c}$ photon energy / time / area

~ $\frac{1}{c} \cos\Theta \frac{dE_\nu}{dt \, d\nu \, |d\vec{f}|}$

~ $\frac{1}{c} \cos\Theta \; I_\nu \cos\Theta \, d\omega$

$$p_\nu = \frac{1}{c} \int I_\nu \cos^2\Theta \, d\omega$$

$$p_\nu = \frac{4\pi}{c} K_\nu$$

Isotropic radiation field:

$J_\nu = I_\nu$; $F_\nu = 0$; $K_\nu = \frac{1}{3} I_\nu$; $p_\nu = \frac{1}{3} u_\nu$

If ν-dependence not relevant: integration

$$I = \int I_\nu \, d\nu$$

For φ-independence, common definition

$$\mu = \cos\Theta$$

e.g.: $K_\nu = \frac{1}{4\pi} \int_0^\pi I_\nu \cos^2\Theta \; \sin\Theta \, d\Theta \, d\varphi$

$$= \frac{1}{2} \int_{-1}^{+1} I_\nu(\mu) \mu^2 \, d\mu$$

Interaction matter – radiation field via
scattering
absorption
emission

Change dI_ν along ds:

$$dI_\nu = \left.\frac{dI_\nu}{ds}\right|_{\text{absorption}} ds + \left.\frac{dI_\nu}{ds}\right|_{\text{scattering}} ds + \left.\frac{dI_\nu}{ds}\right|_{\text{emission}} ds$$

$$\left.\frac{dI_\nu}{ds}\right|_{\text{absorption}} = -I_\nu k_\nu^a$$

k_ν^a : volume opacity (cross section / volume)

$$\left.\frac{dI_\nu}{ds}\right|_{\text{emission}} = j_\nu' \quad \text{emissivity}$$

scattering: contributes both to opacity and emissivity formally

Including scattering in k_ν and j_ν:

$$\frac{dI_\nu}{ds} = -I_\nu k_\nu + j_\nu$$

specific opacity κ_ν:

$$\kappa_\nu = k_\nu / \rho \qquad \text{(cross section / mass)}$$

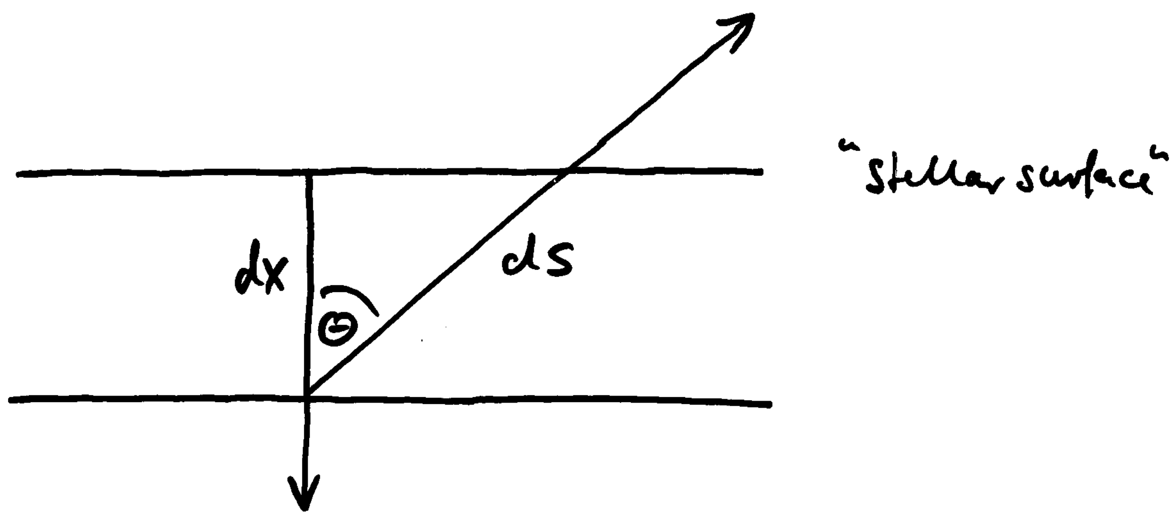
Definition: optical depth τ_ν

$$d\tau_\nu = k_\nu \, ds$$

$\hat{=}$ ratio geometrical distance to mean free path

So far, $ds \parallel dI_\nu$ assumed implicitly

Consider now



$$-dx = \cos\Theta \, ds$$

$$-\cos\Theta \, \frac{dI_\nu}{dx} = -k_\nu I_\nu + j_\nu$$

Equation for radiation transport:

$$\cos\Theta \, \frac{dI_\nu}{k_\nu dx} = I_\nu - \frac{j_\nu}{k_\nu}$$

$$\mu \, \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

with

$\mu = \cos\Theta$; $S_\nu = \frac{j_\nu}{k_\nu}$ source function

$d\tau_\nu = k_\nu \, dx$ optical depth

Radiation transport using moments of the radiation field

aim: replacement of angular dependence

$$\cos\Theta \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

0^{th} order moment: $\frac{1}{4\pi}\int d\omega$

$$\frac{d}{d\tau_\nu} \frac{1}{4\pi} F_\nu = J_\nu - \frac{1}{4\pi}\int S_\nu d\omega$$

S_ν isotropic:

$$\frac{1}{4\pi}\frac{dF_\nu}{d\tau_\nu} = J_\nu - S_\nu$$

1^{st} order moment: $\frac{1}{4\pi}\int \cos\Theta \, d\omega$

$$\frac{dK_\nu}{d\tau_\nu} = \frac{1}{4\pi} F_\nu - \frac{1}{4\pi}\int \cos\Theta \, S_\nu \, d\omega$$

S_ν isotropic:

$$\frac{dK_\nu}{d\tau_\nu} = \frac{1}{4\pi} F_\nu$$

$$\frac{dp_\nu}{d\tau_\nu} = \frac{1}{c} F_\nu$$

Stellar interiors: Frequency dependence ignored, ν-integration

ν - integration

$$F = \int F_\nu \, d\nu \qquad p_{rad} = \int p_\nu \, d\nu$$

$$F_\nu = - \frac{c}{k_\nu} \frac{dp_\nu}{dr} \quad ; \quad F = - \frac{c}{\bar{k}} \frac{dp_{rad}}{dr}$$

Relation $\quad k_\nu \longleftrightarrow \bar{k} \quad$?

Definition of the mean $\bar{k}$?

Local thermodynamic equilibrium
LTE

Assumptions:

1) Local isotropy of radiation field

2) Source function S_ν corresponds to that of black body $B_\nu(T)$

System big enough for thermodynamic equilibrium and definition of temperature
small enough for isotropy to be reasonable approximation in "certain" respects

Globally: No thermodyn. equilibrium
temperature ?
flux does not vanish

Consequences:

$$S_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1}$$

$$B(T) = \int B_\nu d\nu = \frac{\sigma}{\pi} T^4 \quad ; \quad \sigma = \frac{ac}{4}$$

S_ν isotropic and $\frac{dF_\nu}{d\tau_\nu} = 0$ $\left[F_\nu \neq 0 \text{ globally} \right]$

$$J_\nu = S_\nu ; \quad J_\nu = I_\nu ; \quad \frac{dK_\nu}{d\tau_\nu} = \frac{1}{4\pi} F_\nu$$

$$\frac{dp_\nu}{d\tau_\nu} = \frac{1}{c} F_\nu$$

$$I_\nu = S_\nu = B_\nu(T)$$

$$p_\nu = \frac{4\pi}{c} K_\nu = \frac{4\pi}{c}\frac{1}{3} I_\nu = \frac{4\pi}{3c} B_\nu$$

$$u_\nu = 3 p_\nu = \frac{4\pi}{c} I_\nu$$

$$p_{rad} = \frac{a}{3} T^4 \quad ; \quad u_{rad} = a T^4$$

Opacity mean

$$F = \int F_\nu d\nu = - \int d\nu \frac{c}{k_\nu} \frac{dp_\nu}{dr}$$

$$= - \int d\nu \frac{c}{k_\nu} \frac{4\pi}{3c} \frac{dB_\nu}{dT} \frac{dT}{dr}$$

$$\frac{dp_{rad}}{dr} = \int d\nu \frac{dp_\nu}{dr} = \int d\nu \frac{4\pi}{3c} \frac{dB_\nu}{dT} \frac{dT}{dr}$$

$$F \overset{!}{=} - \frac{c}{\bar{k}} \frac{dp_{rad}}{dr}$$

$$\frac{1}{\bar{k}} = \frac{\int_0^\infty d\nu \frac{1}{k_\nu} \frac{dB_\nu}{dT}}{\int_0^\infty d\nu \frac{dB_\nu}{dT}}$$

$\bar{k}$: Rosseland mean of opacity

$$F = - \frac{4ac}{3} T^3 \frac{1}{\bar{k}} \frac{dT}{dr}$$

Diffusion equation for radiation transport

"Diffusion approximation" [stellar interiors]

Definition: Luminosity L

$$L(r) = 4\pi r^2 F$$

$$L(r) = -\frac{16\pi a c}{3\bar{k}} r^2 T^3 \frac{dT}{dr}$$

Alternative forms:

$$F = -\frac{1}{3}\frac{c}{\bar{k}}\frac{du_{rad}}{dr}$$

$$-\frac{dp_{rad}}{dr} = \frac{\bar{k}\, L(r)}{4\pi r^2 c}$$

Radiative acceleration:

$$a_{rad} = \frac{\bar{k}/\varrho\, L(r)}{4\pi r^2 c}$$

Requirement for static star:

$$a_{rad}(R) < g(R)$$

g: gravity $\quad g(R) = \frac{GM}{R^2}$

$\Rightarrow$ upper limit for luminosity

$$L(R) < \frac{4\pi c G M}{\bar{k}/\varrho} = L_{edd}$$

L_{edd} : Eddington luminosity

Properties:

$$L_{edd} \sim M \quad ; \quad \sim \frac{1}{\bar{\kappa}/\rho}$$

$$M = M_{\odot} \qquad L_{edd} \approx 10^4 L_{\odot}$$

Grey approximation

Assumption : $\frac{\partial k_\nu}{\partial \nu} = 0$

$\tau_\nu = \tau$ independent of ν

simple but not realistic

Integration $\int d\nu$ over all equations and quantities : $F = \int F_\nu d\nu$ .....

Consider

$$\mu \frac{dI}{d\tau} = I - S$$

$$\frac{1}{4\pi} \frac{dF}{d\tau} = J - S$$

Radiative equilibrium: $\frac{dF}{d\tau} = 0$

"energy conservation"

$$J = S$$

$$\mu \frac{dI}{d\tau} = I - J = I - \frac{1}{4\pi} \int I d\omega$$

Eddington's equation for grey transport

no LTE, no isotropy for I

Assumption $\frac{\partial I_\nu}{\partial \varphi} = 0$

axisymmetry

Power series solution for $I(\tau, \Theta)$

$$I(\tau,\Theta) = I_0(\tau) + I_1(\tau)\cos\Theta$$

Consequences

$$F = \frac{1}{\pi}\int I \cos\Theta \, d\omega = \frac{I_1}{\pi}\int \cos^2\Theta \, d\omega = \frac{I_1}{\pi}\,\frac{4\pi}{3}$$

$$\cos\Theta \frac{dI_0}{d\tau} + \cos^2\Theta \frac{dI_1}{d\tau} = I_0 + I_1\cos\Theta - I_0 \underbrace{\int \frac{d\omega}{4\pi}}_{1} - I_1 \underbrace{\int \cos\Theta \frac{d\omega}{4\pi}}_{0}$$

$$\frac{dI_1}{d\tau} = 0 \quad \Rightarrow \quad I_1, F \text{ constant} \; ; \; I_1 = \frac{3}{4}F$$

$$\frac{dI_0}{d\tau} = I_1 \Rightarrow I_0 = I_1\tau + \text{constant} = \frac{3}{4}F\tau + \text{constant}$$

$$I(\tau,\Theta) = \frac{3}{4}F\tau + \text{constant} + \frac{3}{4}F\cos\Theta$$

Boundary condition: no incoming radiation at the "surface"

$$F^-(\tau=0) = 0$$

$$F^-(\tau=0) = \frac{1}{\pi} \int_{\pi/2}^{\pi} d\Theta \int_0^{2\pi} d\varphi \; I(0,\Theta) \cos\Theta \, d\omega$$

$$= \frac{\text{constant}}{\pi} \underbrace{\iint \cos\Theta \, d\omega}_{-\pi} + \frac{3}{4\pi} F \underbrace{\iint \cos^2\Theta \, d\omega}_{2/3\,\pi}$$

$$F^-(\tau=0) = 0 \quad \Rightarrow \quad \text{constant} = F/2$$

$$I(\tau,\Theta) = \frac{F}{2}\left(1 + \frac{3}{2}\tau + \frac{3}{2}\cos\Theta\right)$$

$\tau \gg 1$ isotropic radiation field

$\tau \ll 1$ anisotropy large

Application: limb darkening

$$\frac{I(0,\Theta)}{I(0,0)} = \frac{2}{5}\left(1 + \frac{3}{2}\cos\Theta\right)$$

Grey transport and LTE

Assumptions:

1) Isotropy $I = \mathcal{J}$

2) $I_\nu = B_\nu$; $I = B = \frac{\sigma}{\pi} T^4$

Anisotropy: temperature cannot be defined

$\Rightarrow$ Use $\mathcal{J}$ instead of I for definition

$$\mathcal{J} = I = \frac{\sigma}{\pi} T^4$$

$$\mathcal{J} = \int I \frac{d\omega}{4\pi} = \frac{F}{2}\left(1 + \frac{3}{2}\tau\right) = \frac{\sigma}{\pi} T^4$$

Temperature for $\tau = 0$:

$$\sigma T^4(\tau = 0) = \pi F/2$$

Stefan-Boltzmann law, definition of effective Temperature:

$$\sigma T_{eff}^4 = \pi F$$

$$\Rightarrow \quad T^4(\tau = 0) = \frac{1}{2} T_{eff}^4$$

Temperature stratification of the grey atmosphere

$$T^4(\tau) = T^4(\tau=0)\left(1+\frac{3}{2}\tau\right)$$
$$= \frac{1}{2}T_{eff}^4\left(1+\frac{3}{2}\tau\right)$$

$T = T_{eff}$ for $\tau = 2/3$

$\tau = 2/3$: Photosphere

Stellar interior : $\tau \geq 2/3$

Stellar interiors

- Consider $\tau \geq 2/3$
- Radiation field isotropic
- Diffusion approximation for radiation transport

$$\vec{F} = - \frac{4ac}{3} \frac{T^3}{\bar{\kappa}\rho} \nabla T$$

$\bar{\kappa}$: specific opacity (Rosseland) $\bar{\kappa} = \bar{k}/\rho$

$$L(r) = - \frac{16\pi ac}{3\bar{\kappa}\rho} r^2 T^3 \frac{dT}{dr}$$

- Outer boundary: Photosphere $\tau = 2/3$
- Boundary condition: Stefan-Boltzmann

$$L = 4\pi R^2 \sigma T_{eff}^4$$

Stellar parameters

- Luminosity L
  absolute bolometric magnitude M_{bol}

- Effective temperature
  colour, e.g. $B-V$
  spectral type

- Mass

Hertzsprung-Russell-diagram
HRD

$$L \longleftrightarrow T_{eff}$$

Stefan-Boltzmann:

$$L = 4\pi R^2 \sigma T_{eff}^4$$

- Main sequence
- Giants
- White dwarfs

HRD of star clusters: Isochrones

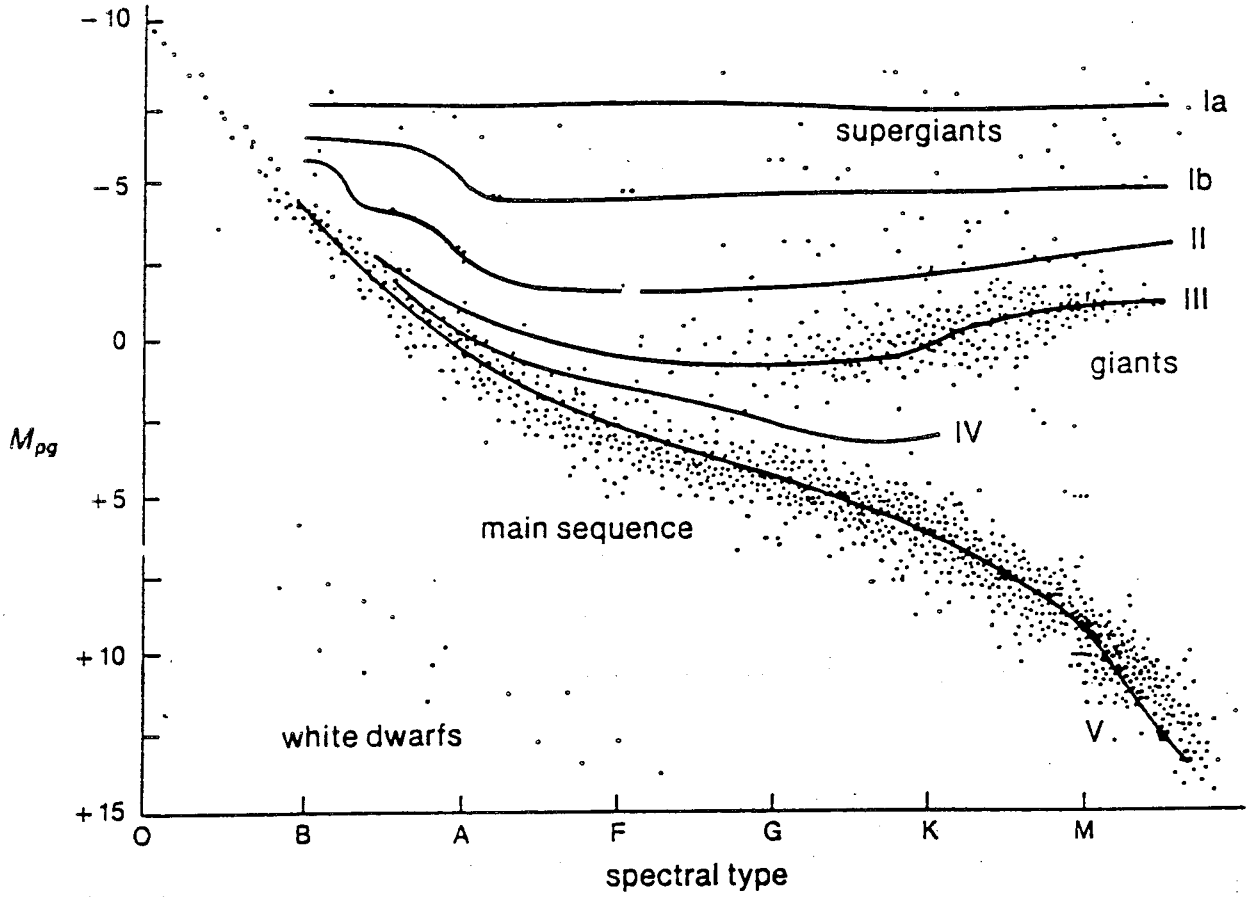


Figure 3.2. Hertzsprung-Russell diagram for stars in solar vicinity, showing absolute photographic magnitude versus spectral type. Solid lines are luminosity classes (approximate).

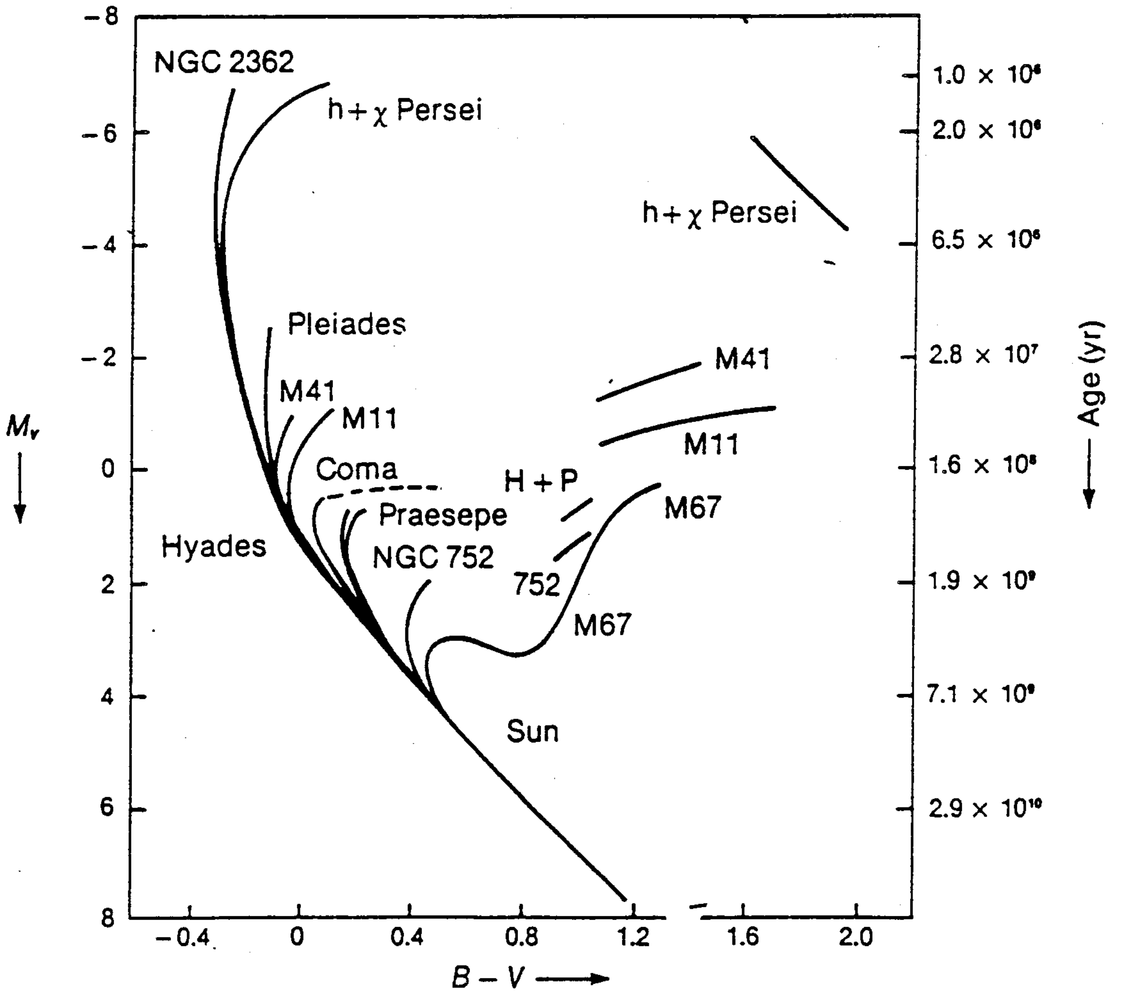


Figure 3.5. Color-magnitude diagram for selected galactic clusters (Population I).

Stellar structure

Definition: Star = Gaseous configuration in hydrostatic equilibrium of pressure gradients and gravitation

First approximation: Spherical symmetry
No rotation
No magnetic fields
Single objects

Description

Euler	Lagrange
r, t independent variables	r_0, t ; $r_0(t)$ indep. var.
	r_0 : initial position of fixed mass element

Lagrange: Consider

$$M_r = \int_0^r 4\pi\rho r^2 dr \qquad M_r \in [0, M]$$

the mass M_r inside a given radius r as independent variable rather than r.

$r(M_r)$:

$$\frac{\partial r}{\partial M_r} = \frac{1}{4\pi\rho r^2}$$

Gravitational Potential ϕ

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 4\pi G\rho$$

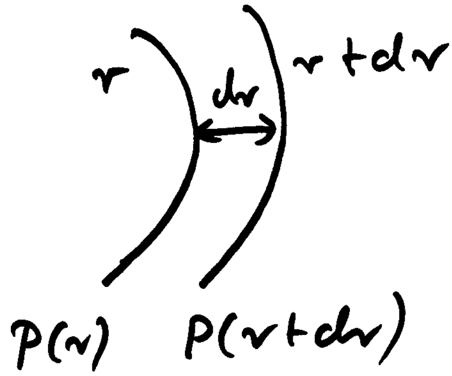
$$r^2 \frac{\partial \phi}{\partial r} = G \int_0^r 4\pi\rho r^2 dr = G M_r$$

Gravity g:

$$g = -\frac{\partial \phi}{\partial r} = -\frac{G M_r}{r^2}$$

Hydrostatic equilibrium

Consider spherical shell



p : pressure

Force exerted on shell:

$$\left[p(r+dr) - p(r)\right] 4\pi r^2 = \frac{dp}{dr}\, dr\, 4\pi r^2$$

Gravitational force on shell,

$$g\, dm = g \rho\, 4\pi r^2 dr$$

Hydrostatic equilibrium:

$$\frac{\partial p}{\partial r} = g \rho$$

$$\frac{1}{\rho} \frac{\partial p}{\partial r} = g = - \frac{G M_r}{r^2}$$

Transformation for M_r:

$$\frac{\partial p}{\partial M_r} = - \frac{G M_r}{4\pi r^4}$$

Boundary condition:

$$p(M_r = M) = 0$$

Dynamical Phases

Violation of equilibrium leads to acceleration

Newton's law:

$$\frac{\partial^2 r}{\partial t^2} = - \frac{1}{\rho} \frac{\partial p}{\partial r} - \frac{G M_r}{r^2}$$

Transformation to M_r:

$$\frac{\partial p}{\partial M_r} = - \frac{G M_r}{4\pi r^4} - \frac{1}{4\pi r^2} \frac{\partial^2 r}{\partial t^2}$$

Virial theorem

Assumption: Pressure given by equation of state of ideal gas:

$$P = \frac{R}{\mu} \rho T \quad ; \quad c_V = \frac{3}{2} \frac{R}{\mu}$$

Thermal energy

$$E_T = \int_0^M c_V T \, dM_r = \int_0^M \frac{3}{2} \frac{R}{\mu} T \, dM_r$$

Potential energy of mass element at r:

$$dE_G = dM_r \int_r^\infty - \frac{GM_r}{r'^2} dr' = - \frac{GM_r}{r} dM_r$$

Total potential energy:

$$E_G = - \int_0^M \frac{GM_r}{r} dM_r$$

Consider Mechanical equilibrium

$$\frac{dp}{dM_r} = - \frac{GM_r}{4\pi r^4} \qquad \times 4\pi r^3 dM_r \; ; \int_0^M$$

$$\int_0^M 4\pi r^3 \frac{dp}{dM_r} dM_r = -\int_0^M \frac{GM_r}{r} dM_r = E_G$$

$$= \left[4\pi r^3 p\right]_0^M - \int_0^M 12\pi r^2 \frac{dr}{dM_r} p \, dM_r$$

$$= -3\int_0^M p/\rho \, dM_r = -3\int_0^M \frac{\mathcal{R}}{\mu} T \, dM_r = -2E_T$$

Virial theorem:

$$E_G = -\underset{\substack{\uparrow \\ \text{depends on equation of state}}}{2} E_T$$

Total energy:

$$E_{tot} = E_G + E_T = -E_T \underset{\substack{\uparrow \\ \text{(Gravitationally) bound system}}}{<} 0$$

Virial theorem $\neq$ Energy conservation !!

Virial theorem consequence of hydrostatic equilibrium

Contraction of a star:

$$-\delta E_G > 0 \quad \Rightarrow \quad \delta E_T = -\frac{1}{2}\,\delta E_G > 0$$

Star becomes "hotter"

$$\delta E_{tot} = -\,\delta E_T = \frac{1}{2}\,\delta E_G < 0$$

Total energy decreases
due to energy loss
by radiation

$\delta E_{tot} < 0$; $\delta E_T > 0$: negative specific heat

Time scale for thermal evolution:
Kelvin-Helmholtz-time scale τ_{KH}

$$|E_T| \sim |E_G| \sim \frac{GM^2}{R}$$

$$\tau_{KH} \sim \frac{|E_T|}{L} \sim \frac{|E_G|}{L} \sim \frac{GM^2}{RL}$$

sun: $L_\odot = 4\cdot 10^{33}$ erg/sec
$M_\odot = 2\cdot 10^{33}$ g
$R_\odot = 7\cdot 10^{10}$ cm
$\tau_{KH} \sim 3\cdot 10^{7}$ years $< \tau_{Earth} \approx 4\cdot 10^{9}$ years

$\Rightarrow$ Gravitational energy is not the source of stellar radiation

Energy conservation

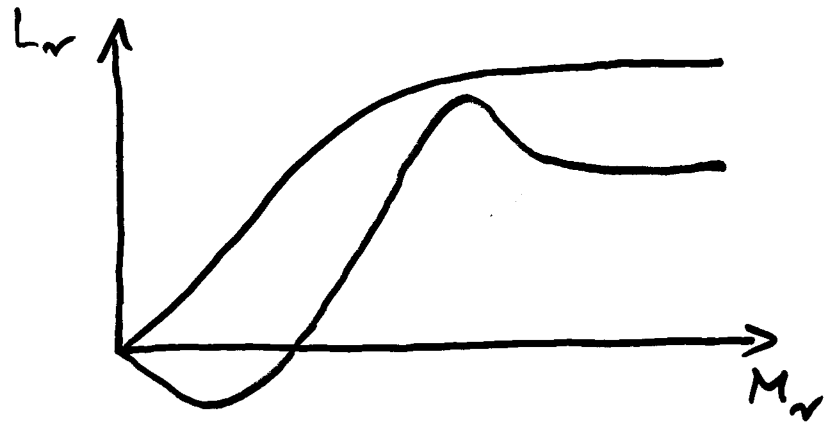
L_r: energy / time flowing through spherical shell at radius r

Boundary conditions:

$$L(M_r=0)=0 \quad ; \quad L_r(M_r=M)=L$$

observed luminosity

$L_r(M_r)$ non-monotonous:



Change of L_r per mass interval:

$$\frac{\partial L}{\partial M_r} = \varepsilon_N - \varepsilon_\nu + \varepsilon_g$$

nuclear energy generation — neutrino losses — change of heat content

$$\varepsilon_g = -\frac{dQ}{dt} = -T\frac{ds}{dt}$$

$$\left.\begin{array}{l}\text{thermal equilibrium}\\ \text{adiabatic changes of state}\end{array}\right\} \quad \varepsilon_g = 0$$

$$\varepsilon_g = -T\frac{ds}{dt}$$

$$s = s(\rho, T) = s(p, T) = \dots$$

for $s = s(p, T)$:

$$\varepsilon_g = -c_p \frac{dT}{dt} - T \frac{\partial s}{\partial p}\Big|_T \frac{dp}{dt}$$

1st law of thermodynamics

$$dQ = du - p/\rho^2 d\rho$$

$$u = u(p, T) \; ; \; \rho = \rho(p, T); \; s = s(p, T)$$

$$dQ = \left(\frac{\partial u}{\partial T}\Big|_p - p/\rho^2 \frac{\partial \rho}{\partial T}\Big|_p\right) dT + \left(\frac{\partial u}{\partial p}\Big|_T - p/\rho^2 \frac{\partial \rho}{\partial p}\Big|_T\right) dp$$

$$= T ds = T \frac{\partial s}{\partial T}\Big|_p dT + T \frac{\partial s}{\partial p}\Big|_T dp$$

$$T \frac{\partial s}{\partial T}\Big|_p = \frac{\partial u}{\partial T}\Big|_p - p/\rho^2 \frac{\partial \rho}{\partial T}\Big|_p \quad (1)$$

$$T \frac{\partial s}{\partial p}\Big|_T = \frac{\partial u}{\partial p}\Big|_T - p/\rho^2 \frac{\partial \rho}{\partial p}\Big|_T \quad (2)$$

$$\frac{\partial}{\partial p}\Big|_T (1) - \frac{\partial}{\partial T}\Big|_p (2):$$

$$-\frac{\partial s}{\partial p}\Big|_T = -\frac{\partial \rho}{\partial T}\Big|_p \left(\frac{1}{\rho^2} - p\frac{2}{\rho^3}\frac{\partial \rho}{\partial p}\Big|_T\right)$$

$$+ \frac{\partial \rho}{\partial p}\Big|_T \left(-p\frac{2}{\rho^3}\frac{\partial \rho}{\partial T}\Big|_p\right) = -\frac{\partial \rho}{\partial T}\Big|_p \frac{1}{\rho^2}$$

$$\varepsilon_g = - c_p \frac{dT}{dt} - \frac{T}{\rho^2} \frac{\partial \rho}{\partial T}\Big|_p \frac{dp}{dt}$$

Definition

$$\delta = - \frac{\partial \log \rho}{\partial \log T}\Big|_p = - \frac{T}{\rho} \frac{\partial \rho}{\partial T}\Big|_p$$

$$\varepsilon_g = - c_p \frac{dT}{dt} + \frac{\delta}{\rho} \frac{dp}{dt}$$

1) Mechanical (dynamical, free-fall) timescale

$$\frac{\partial^2 r}{\partial t^2} \sim \frac{GM_r}{r^2} \,;\, \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{R}{\tau_{Dyn}^2} \sim \frac{GM}{R^2}$$

$$\tau_{Dyn} \sim \left(\frac{R^3}{GM}\right)^{1/2} \sim \left(G\langle\rho\rangle\right)^{-1/2}$$

examples: τ_{Dyn}

sun ~ 25 min

giant $R \sim 10^2 R_\odot$ ~ 20 d

White dwarf $R \sim 10^{-2} R_\odot$ ~ 1.5 sec

Neutron star $R \sim 10^{-5} R_\odot$ $\sim 10^{-4}$ sec

2) Thermal (KH) timescale

$$\tau_{KH} = \frac{GM^2}{RL}$$

3) Nuclear timescale

$$\tau_{Nuc} = \frac{E_{nuc}}{L} = \left(\frac{\Delta M}{M}\right)_{nuc} \frac{Mc^2}{L}$$

$$\left(\frac{\Delta M}{M}\right)_{nuc} \approx 7 \cdot 10^{-3} \text{ at maximum}$$

Sun: $\tau_{nuc} \sim 10^{10}$ years

Relation

$$\tau_{nuc} \gg \tau_{KH} \gg \tau_{Dyn}$$

$\Rightarrow$ Hydrostatic equilibrium satisfied for stellar evolution

but: dynamical phases at certain stages of evolution

Energy transport in stars

Transport by diffusion of radiation:

$$L_r(r) = - \frac{16\pi a c}{3\bar{\kappa}\rho} r^2 T^3 \frac{dT}{dr}$$

$\bar{\kappa}$: specific opacity (Rosseland mean)

Conductive transport

not relevant in ordinary stars

photon mean free path $\gg$ particle (electron) mean free path

degenerate stars: situation may be reversed

Description of conduction as diffusion process:

$$L_{\text{conductive}} = - D_{\text{conductive}} \frac{dT}{dr}$$

For convenience, define conductive opacity κ_{cond} such that radiation diffusion equation remains valid with total opacity κ_{tot}:

$$\frac{1}{\kappa_{\text{tot}}} = \frac{1}{\bar{\kappa}} + \frac{1}{\kappa_{\text{cond}}}$$

Transformation of variables

$$\frac{\partial T}{\partial M_r} = -\frac{3}{64\pi^2 ac}\,\frac{\bar{\kappa}}{r^4 T^3}\,L_r$$

$$=: \nabla\,\frac{T}{P}\,\frac{\partial P}{\partial M_r}$$

Definition of ∇

$$\nabla = \nabla_{rad} = \frac{3}{16\pi ac}\,\frac{\bar{\kappa}\,L_r\,P}{T^4 G M_r}$$

General treatment of energy transport by

$$\frac{\partial T}{\partial M_r} = \nabla\,\frac{T}{P}\,\frac{\partial P}{\partial M_r}$$

Details of transport determine ∇

e.g., radiative transport $\quad \nabla = \nabla_{rad}$

conductive transport $\quad \nabla = \nabla_{rad}$

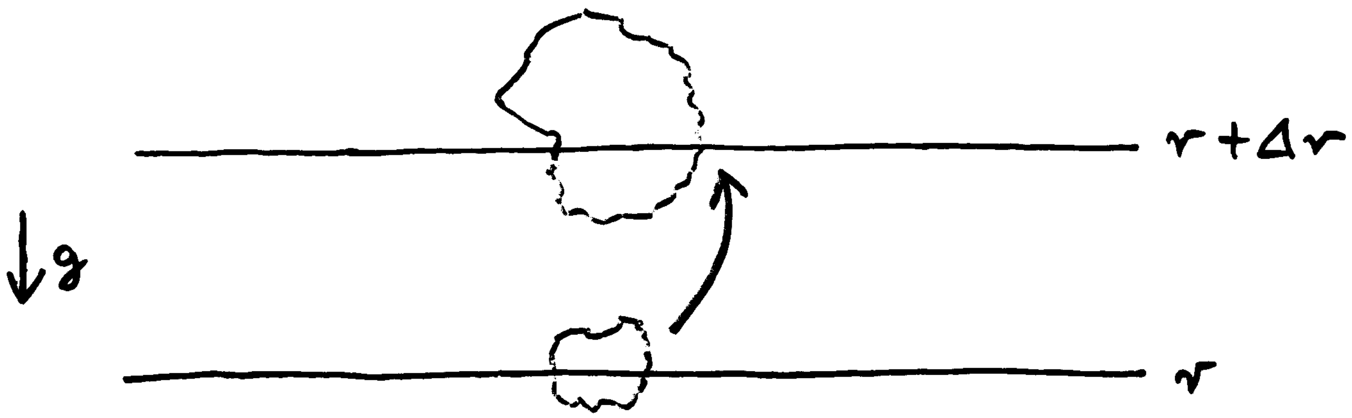
with $\dfrac{1}{\kappa_{tot}} = \dfrac{1}{\bar{\kappa}} + \dfrac{1}{\kappa_{cond}}$

Radiatively determined temperature gradient too steep $\longrightarrow$ instability of stratification

$\longrightarrow$ "Convection"

$\longrightarrow$ energy transport by convection

Stability of Stratification
Criteria for the onset of convection



Buoyancy force:

$$F = -g\,\Delta\rho$$

$$\Delta\rho = \rho_{\text{Element}} - \rho_{\text{surroundings}} = \Delta\rho_E - \Delta\rho_S$$

$$\Delta\rho_S = \frac{d\rho}{dr}\,\Delta r$$

Assumptions: 1) Pressure equilibrium

$$\Delta p_{\text{Element}} = \Delta p_{\text{surroundings}}$$

$$p_{\text{Element}} = p_{\text{surroundings}}$$

2) Adiabatic changes of state for element, displacement sufficiently fast; $p_E = p_E(\rho_E)$

$$\Delta p_{\text{Element}} = \underbrace{\left.\frac{dp}{d\rho}\right|_{ad}}_{=\gamma_{ad}\,p/\rho} \Delta\rho_{\text{Element}}$$

$$\gamma_{ad} = \left.\frac{\partial \log p}{\partial \log \rho}\right|_S \qquad S: \text{entropy}$$

$$\gamma_{ad}\, P/\rho\, \Delta\rho_E = \Delta p_E = \Delta p_S = \frac{dp}{dr}\,\Delta r$$

$$F = -g\left(\frac{\rho}{\gamma_{ad} p}\frac{dp}{dr} - \frac{d\rho}{dr}\right)\Delta r$$

$$F = g\,\Delta r\,\underbrace{\left(\frac{d\rho}{dr} - \frac{\rho}{\gamma_{ad} p}\frac{dp}{dr}\right)}$$

< 0 : stability

> 0 : instability

One formulation of criterion for convective instability

Relation to Rayleigh - Taylor instability

Assumption : Fluid incompressible

$$\Delta\rho_E = 0$$

$$\Rightarrow \Delta\rho = -\Delta\rho_S$$

$$F = g\,\Delta r\,\frac{d\rho}{dr}$$

RT-instability and convective instability rely on the same physics with different properties of matter

$\Rightarrow$ Criteria must not be used simultaneously!

Schwarzschild - Criterion
Ledoux - Criterion

General stability criterion:

$$Q := \frac{d\rho}{dr} - \frac{\rho}{\gamma_{ad} P} \frac{dP}{dr} \quad \lessgtr 0$$ stability / instability

Equation of state

$$\rho = \rho(P, T, \mu)$$ μ: molecular weight, chemical composition

Differential form:

$$\frac{1}{\rho}\frac{d\rho}{dr} = \alpha \frac{1}{P}\frac{dP}{dr} - \delta \frac{1}{T}\frac{dT}{dr} + \varphi \frac{1}{\mu}\frac{d\mu}{dr}$$

$$\alpha = \left.\frac{\partial \log \rho}{\partial \log P}\right|_{T,\mu} \qquad \delta = -\left.\frac{\partial \log \rho}{\partial \log T}\right|_{P,\mu}$$

$$\varphi = \left.\frac{\partial \log \rho}{\partial \log \mu}\right|_{T,P}$$

$$Q = \rho \frac{1}{P}\frac{dP}{dr}\,\delta\left(\frac{1}{\delta}\left(\alpha - \frac{1}{\gamma_{ad}}\right) - \underbrace{\frac{d\log T}{d\log P}}_{\nabla} + \frac{\varphi}{\delta}\underbrace{\frac{d\log\mu}{d\log P}}_{\nabla_\mu}\right)$$

$$\nabla \lessgtr \underbrace{\frac{1}{\delta}\left(\alpha - \frac{1}{\gamma_{ad}}\right)}_{?} + \frac{\varphi}{\delta}\nabla_\mu$$ stability / instability

Radiative transport: $\nabla = \nabla_{rad}$

$$\left.\begin{aligned} \gamma_{ad} &= \left.\frac{\partial \log P}{\partial \log \rho}\right|_s \\ \nabla_{ad} &= \left.\frac{\partial \log T}{\partial \log P}\right|_s \end{aligned}\right\}$$ entirely determined by equation of state

$$\frac{1}{\gamma_{ad}} = \left.\frac{\partial \log \rho}{\partial \log P}\right|_s = \alpha \underbrace{\left.\frac{\partial \log P}{\partial \log P}\right|_s}_{=1} - \delta \underbrace{\left.\frac{\partial \log T}{\partial \log P}\right|_s}_{=\nabla_{ad}}$$

$$\frac{1}{\gamma_{ad}} = \alpha - \delta \nabla_{ad}$$

Stability criterion:

$$\nabla_{rad} \lesseqgtr \nabla_{ad} + \frac{\varphi}{\delta} \nabla_\mu$$ stability / instability

$\nabla_\mu \neq 0$: Ledoux-criterion
$\nabla_\mu = 0$: Schwarzschild-criterion
} for convective instability

<u>Remark</u>: Most general form of criterion may be written as

$$\frac{ds}{dr} \gtrless 0$$ stability / instability

Physical origin: buoyancy

Treatment of Convection and convective transport

$\nexists$ consistent theory of convection !!

Problem: calculate ∇

Limiting case: convection effective (stellar cores)
Consequence: reduces ∇
Marginally stable state:

$$\nabla = \nabla_{ad}$$

Convection adiabatic

General case: convection inefficient
non-adiabatic

$$\nabla_{ad} < \nabla < \nabla_{rad}$$

"Theory" for non-adiabatic convection:
Mixing-length-theory MLT (Prandtl, Böhm-Vitense)

Parameter of MLT: Mixing length

$$l = \alpha \underbrace{\left(\frac{1}{p}\frac{\partial p}{\partial r}\right)^{-1}}_{:= -H_p \text{ pressure scale height}}$$

$$\alpha \sim O(1) \qquad \alpha = 1 \dots 2$$

Result of MLT: ∇_{MLT} : $\nabla_{ad} < \nabla_{MLT} < \nabla_{rad}$

Mixing length theory MLT

In convective region total flux consists of radiative and convective part

$$F_{tot} = F_{rad} + F_{con}$$

Definition of ∇_{rad}:

$$F = \frac{4ac}{3} \frac{T^4}{\kappa \rho H_p} \nabla_{rad} \qquad (1)$$

Definition of ∇:

$$F_{rad} = \frac{4ac}{3} \frac{T^4}{\kappa \rho H_p} \nabla \qquad (2)$$

Energy flux by moving (convective) mass element

$$F'_{con} = \underbrace{c_p \Delta T}_{} \rho v$$

$\Delta p = 0$: pressure equilibrium

$$\Delta T = T_{element} - T_{surrounding}$$

Total convective flux : suitable average

$$F_{con} = \rho \bar{v} c_p \overline{\Delta T} \qquad \bar{v}, \overline{\Delta T} = ?$$

Parameter of MLT: Mixing length l corresponds to mean free path of convection element

r_0: $\Delta T(r_0) = 0$; $\Delta \rho(r_0) = 0$

Taylor-expansion around r_0:

$$\Delta T = \Delta T(r_0) + \frac{d\,\Delta T}{dr}(r - r_0)$$

$$= \left(\frac{dT_{elem}}{dr} - \frac{dT_{surr}}{dr}\right)(r - r_0)$$

$$= -\overline{T}\,\frac{1}{H_p}\Bigg(\underbrace{\frac{\partial \log T_e}{\partial \log P}}_{:=\nabla_e} - \underbrace{\frac{\partial \log T_s}{\partial \log P}}_{=\nabla}\Bigg)(r - r_0)$$

$$\Delta \rho = \Delta \rho(r_0) + \frac{d\,\Delta \rho}{dr}(r - r_0)$$

$$= \left(\frac{d\rho_e}{dr} - \frac{d\rho_s}{dr}\right)(r - r_0)$$

$$= -\overline{\rho}\,\frac{1}{H_p}\left(\frac{\partial \log \rho_e}{\partial \log P} - \frac{\partial \log \rho_s}{\partial \log P}\right)(r - r_0)$$

Equation of state:

$$\Delta \rho = -\overline{\rho}\,\frac{1}{H_p}\left(\alpha - \delta\nabla_e - \alpha + \delta\nabla\right)(r - r_0)$$

Identify $\overline{T} = T$; $\overline{\rho} = \rho$

$$\Delta T = -\frac{T}{H_p}(\nabla_e - \nabla)(r - r_0)$$

$$\Delta \rho = \frac{\rho\delta}{H_p}(\nabla_e - \nabla)(r - r_0)$$

Change of potential energy for the rising element:

$$\Delta E = \int_r^{r_0} g\,\Delta\rho\,dr = \frac{g\rho\delta}{H_p}(\nabla_e - \nabla)\int_r^{r_0}(r - r_0)\,dr$$

$$\Delta E = \frac{g\rho\delta}{H_p}(\nabla_e - \nabla)\frac{(r - r_0)^2}{2}$$

Assumption:

$$\Delta E \begin{array}{l} \nearrow \text{Work needed for expansion} \sim \frac{1}{2}\Delta E \\ \\ \searrow \text{Kinetic energy} \sim \frac{1}{2}\Delta E \end{array}$$

$$\Delta E_{kin} = \frac{1}{2}\rho v^2 = \frac{1}{2}\Delta E = \frac{g\rho\delta}{2H_p}(\nabla_e - \nabla)\frac{(r - r_0)^2}{2}$$

Velocity of convective element:

$$v = \left[\frac{g\delta}{2H_p}(\nabla_e - \nabla)\right]^{1/2}(r - r_0)$$

Average of ΔT and v over mixing length l:

$$\bar{v} = \frac{1}{l}\int_{r_0 - l}^{r_0} |v|\,dr \quad ; \quad \overline{\Delta T} = \frac{1}{l}\int_{r_0 - l}^{r_0} |\Delta T|\,dr$$

$$\overline{\Delta T} = \frac{T}{2H_p}(\nabla - \nabla_e)\, l$$

$$\bar{v}^2 = \frac{g\delta}{2H_p}(\bar{\nabla} - \bar{\nabla}_e)\, \frac{l^2}{4} \qquad (3)$$

$$F_{con} = \rho \bar{v}\, c_p\, \overline{\Delta T}$$

$$F_{con} = \rho c_p T (g\delta)^{1/2} \frac{l^2}{4\sqrt{2}\, H_p^{3/2}} (\bar{\nabla} - \bar{\nabla}_e)^{3/2} \qquad (4)$$

$\Rightarrow$ 4 equations for unknowns

F_{con}, F_{rad}, $\bar{\nabla}_e$, $\bar{\nabla}$, $\bar{v}$

5th equation of MLT: Consider energy loss of convective element during motion

Radiative flux:

$$F = -\frac{4ac}{3}\frac{T^3}{k\rho}\frac{dT}{dr}$$

$$\frac{dT}{dr} \sim \frac{\Delta T}{l/2}$$ dimension of convective element $\sim l$

Energy loss per mass and time

$$\dot{Q} = F\,\frac{S}{\rho V}$$

S: surface of convective element
V: volume of convective element

$$\dot{Q} = -\frac{8ac}{3}\frac{T^4}{k\rho^2}\frac{\nabla - \nabla_e}{l H_p}\frac{S}{V}(r - r_0)$$

Total energy loss:

$$\Delta Q = \int \dot{Q}\, dt = \int_r^{r_0} \frac{\dot{Q}}{v}\, dr = \frac{\dot{Q}}{v}(r - r_0)$$

[$\dot{Q}/v$ independent of r]

Average over mixing length:

$$\overline{\Delta Q} = \frac{\dot{Q}}{v}\frac{\ell}{2} = -\frac{2ac}{3}\frac{T^4}{\kappa\rho^2}\frac{\ell}{H_p}\frac{\nabla - \nabla_e}{v}\frac{S}{V}$$

Energy loss in terms of thermodynamic quantities:

$$\Delta Q = T\,\Delta S_{element} = T\left(\left.\frac{\partial S}{\partial p}\right|_T \Delta p_e + \left.\frac{\partial S}{\partial T}\right|_p \Delta T_e\right)$$

$$= \frac{T^2}{p}\Delta p_e \underbrace{\left.\frac{\partial S}{\partial T}\right|_p}_{c_p/T}\left(\underbrace{\frac{p}{T}\left.\frac{\partial S}{\partial p}\right|_T \Big/ \left.\frac{\partial S}{\partial T}\right|_p}_{-\nabla_{ad}} + \underbrace{\frac{p}{T}\frac{\Delta T_e}{\Delta p_e}}_{\nabla_e}\right)$$

$$T_e = T_e(p_e = p_s)\ ; \quad \Delta T_e = \frac{\partial T_e}{\partial p}\Delta p_e$$

$$\left.\frac{\partial S}{\partial p}\right|_T \Big/ \left.\frac{\partial S}{\partial T}\right|_p = \frac{\partial(S,T)}{\partial(p,T)}\frac{\partial(T,p)}{\partial(S,p)} = -\frac{\partial(T,S)}{\partial(p,S)}$$

$$= -\left.\frac{\partial T}{\partial p}\right|_S$$

$$\frac{\partial(u,v)}{\partial(x,y)} := \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}\ ; \quad \frac{\partial(u,y)}{\partial(x,y)} = \left.\frac{\partial u}{\partial x}\right|_y$$

Properties: antisymmetric
in u, v and x, y

$$\frac{\partial(u,v)}{\partial(t,s)} \frac{\partial(t,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$$

$$\Delta Q = c_p T \left(\nabla_e - \nabla_{ad} \right) \frac{\Delta p_e}{p}$$

$$\Delta p_e = \Delta p_s = p_s(r) - p_s(r_0) = \frac{\partial p}{\partial r} (r - r_0)$$

$$\Delta Q = - c_p T \left(\nabla_e - \nabla_{ad} \right) \frac{1}{H_p} (r - r_0)$$

Average over mixing length:

$$\overline{\Delta Q} = - c_p T \left(\nabla_e - \nabla_{ad} \right) \frac{1}{H_p} \ell/2$$

$$\frac{\nabla_e - \nabla_{ad}}{\nabla - \nabla_e} = \frac{4ac}{3} \frac{T^3}{\kappa \rho^2 c_p \bar{v}} \frac{S}{V}$$

S/V with dimension of convective element ℓ:

sphere: $S/V = \frac{3}{\ell}$ (minimum)

cube: $S/V = \frac{6}{\ell}$

adopt $S/V = \frac{9/2}{\ell}$ for standard MLT

Equations of Mixing length theory

$$F_{tot} = F_{rad} + F_{con} = \frac{4ac}{3} \frac{T^4}{\kappa \rho H_p} \nabla_{rad} \quad (1)$$

$$F_{rad} = \frac{4ac}{3} \frac{T^4}{\kappa \rho H_p} \nabla \quad (2)$$

$$F_{con} = \rho c_p T (g\delta)^{1/2} \frac{\ell^2}{4\sqrt{2}\, H_p^{3/2}} (\nabla - \nabla_e)^{3/2} \quad (3)$$

$$\bar{v}^2 = g\delta(\nabla - \nabla_e) \frac{\ell^2}{8 H_p} \quad (4)$$

$$\frac{\nabla_e - \nabla_{ad}}{\nabla - \nabla_e} = \frac{6acT^3}{\kappa \rho^2 c_p \bar{v} \ell} \quad (5)$$

Free parameter: Mixing length ℓ

Relation to typical length scale:

pressure / density scale height

usually:

$\ell = \alpha H_p \qquad \alpha = O(1) = 1 \ldots 2$

Solution of MLT equations with prescribed

$F_{tot}, T, P, \rho, \kappa, H_p, \ell, c_p, g, \delta, \nabla_{ad}, \nabla_{rad}$

unknown:

$F_{rad}, F_{con}, \bar{v}, \nabla, \nabla_e$

Procedure for solution

add (2) and (3) $\Rightarrow$ 1. equation in ∇, ∇_e (A)
insert (4) in (5) $\Rightarrow$ 2. equation in ∇, ∇_e (B)

A, B represent two algebraic equations for ∇, ∇_e

lead to cubic equation with a single real solution, analytically representable

Determine $\nabla, \nabla_e \Rightarrow \bar{v}$ (4), F_{con} (3), F_{rad} (2)

Limiting cases:

Convection efficient $\quad \nabla \to \nabla_{ad} \quad$ stellar core
ρc_p large

Convection inefficient $\quad \nabla \to \nabla_{rad} \quad$ stellar envelope
ρc_p small

MLT usually not used in stellar cores

Convective overshooting:
Convective elements penetrate into radiative, convectively stable region
measured in terms of H_p, value?

Chemical Composition

Mixture of nuclei with mass m_i, number density n_i, total mass density ρ

Mass fraction:

$$X_i := \frac{m_i n_i}{\rho}$$

$$\sum X_i = 1$$

Hydrogen mass fraction : $X_H = X$

Helium " " $X_{He} = Y$

Remaining " " $\sum_{i \neq H, He} X_i = Z$

Remaining nuclei : "Metals" ; "Heavy elements"

$$Z = 1 - X - Y$$

Typical values:

$$X = 0.6 \dots 0.7$$

$$Y = 0.36 \dots 0.3$$

$$Z = 0.04 \dots 0.001$$

Chemical composition function of mass coordinate

$$X_i = X_i(M_r, \underset{\substack{\uparrow \\ \text{evolution}}}{t})$$

Variation of composition with time

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho}\left[\sum_j r_{ji} - \sum_k r_{ik}\right]$$

r_{ji} : reaction rate $j \to i$
r_{ik} : " $i \to k$

r : reactions per volume and time (T, ρ)
(nuclear physics)

Change of composition associated with (nuclear) energy generation

reaction $j \to i$: energy release Q_{ji}

energy generation rate:

$$\varepsilon_{ji} = \frac{Q_{ji}\, r_{ji}}{\rho}$$

Total energy generation rate

$$\varepsilon(\rho, T) = \sum \varepsilon_{ji}$$

Convective regions: instantaneous mixing
composition constant

$$\left.\frac{\partial X_i}{\partial M_r}\right|_{\text{convection}} = 0$$

Differential equations of stellar structure and evolution

$$\frac{\partial r}{\partial M_r} = \frac{1}{4\pi r^2 \rho} \qquad (1)$$

$$\frac{\partial p}{\partial M_r} = - \frac{G M_r}{4\pi r^4} - \frac{1}{4\pi r^2} \frac{\partial^2 r}{\partial t^2} \qquad (2)$$

$$\frac{\partial L_r}{\partial M_r} = \varepsilon_{nuc} - \varepsilon_\nu - c_p \frac{\partial T}{\partial t} + \frac{\delta}{\rho} \frac{\partial p}{\partial t} \qquad (3)$$

$$\frac{\partial T}{\partial M_r} = - \frac{G M_r T}{4\pi r^4 P} \nabla \qquad (4)$$

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho} \left[\sum_j r_{ji} - \sum_k r_{ik} \right] \qquad (5)$$

with $\nabla_{ad} < \nabla = \nabla_{MLT} < \nabla_{rad}$

in convective regions

$\nabla = \nabla_{rad}$

in radiative regions

$$\left.\begin{aligned} \varepsilon_{nuc} &= \varepsilon_{nuc}(p, T, X_i) \\ \varepsilon_\nu &= \varepsilon_\nu(p, T, X_i) \\ r_{ji} &= r_{ji}(p, T, X_i) \end{aligned}\right\} \text{ nuclear physics}$$

$\kappa = \kappa(p, T, X_i)$ atomic physics

Closure of system by prescription of equation of state (EOS):

$$\rho = \rho(p, T, X_i)$$

EOS implies:

$$c_p = c_p(p, T, X_i)$$
$$\delta = \delta(p, T, X_i)$$
$$\nabla_{ad} = \nabla_{ad}(p, T, X_i)$$

Timescales

$$\tau_{nuc} \gg \tau_{KH} \gg \tau_{Dyn}$$

Hydrostatic equilibrium:

$$\frac{\partial^2 r}{\partial t^2} = 0 \quad \text{in momentum equation}$$

Thermal equilibrium:

$$\frac{\partial T}{\partial t} = 0 \quad \frac{\partial p}{\partial t} = 0 \quad \text{in energy equation}$$

Boundary Conditions

Stellar evolution: Mixed initial / boundary value problem

Central boundary conditions:

$$r(M_r = 0) = 0$$
$$L_r(M_r = 0) = 0$$

Surface boundary conditions:

$$p(M_r = M) = 0 \qquad \text{"zero condition"}$$
$$T(M_r = M) = 0 \qquad \text{not realistic}$$

"Surface" of the star: Photosphere $\tau = \tau_{eff} = 2/3$

Stefan-Boltzmann:

$$T^4(M_r=M) = T_{eff}^4 = \frac{L_r(M_r=M)}{4\pi\sigma\, r^2(M_r=M)}$$

For second condition consider hydrostatic atmosphere:

$$\frac{\partial p}{\partial r} = g\rho \quad \text{integration from photosphere to } \infty$$

$$-P_{eff} = -p(M_r=M) = \int_{r(M_r=M)}^{\infty} g\rho\, dr = \bar{g}\int_{r(M_r=M)}^{\infty} \rho\, dr$$

$$d\tau = -\kappa \rho \, dr$$

integration from infinity to photosphere

$$\tau_{eff} = 2/3 = -\int_{\infty}^{r(M_r=M)} \kappa \rho \, dr = -\bar{\kappa} \int_{\infty}^{r(M_r=M)} \rho \, dr$$

$$\frac{2}{3} = \bar{\kappa} \int_{r(M_r=M)}^{\infty} \rho \, dr$$

$$\frac{2}{3} \frac{1}{\bar{\kappa}} = - \frac{P_{eff}}{\bar{g}}$$

Approximation: $\bar{\kappa} = \kappa_{eff} = \kappa(M_r = M)$

$$\bar{g} = g_{eff} = - \frac{GM}{r^2(M_r=M)}$$

Boundary condition:

$$P_{eff} = \frac{2}{3} \frac{GM}{r^2(M_r=M)} \frac{1}{\kappa_{eff}}$$

Quality of approximations?

Results are not very sensitive to precise form of second outer BC

Restricted Problem:
Envelope solution

No nuclear processes, constant (given) chemical composition $\Rightarrow$ L = constant
Hydrostatic and thermal equilibrium
Prescribe L, T_{eff}, M (position in HRD, estimate mass)

Problem reduces to initial value problem for mass conservation
momentum conservation
energy transport

integration inwards from the surface with initial conditions

$M_r = M$: $\quad T = T_{eff}$
$R = r(M_r = M)$
from Stefan Boltzmann
$P = P_{eff}$ 2nd outer BC

Numerical Solution

Hydrostatic equilibrium assumed

$\tau_{Evolution} > \tau_{KH}$: (thermal equilibrium)

Variation of composition decoupled from time independent part

Solve (1)-(4) with X_i fixed $\rightarrow$ new $P, T, \ldots$

$\downarrow$

Solve (5) with $P, T, \ldots$ fixed $\rightarrow$ new X_i

$\tau_{Evolution} \approx \tau_{KH}$

Treatment of (1)-(4) / (5) as before but

Solve (1)-(4) with ε_g fixed $\rightarrow$ new $P, T \ldots$

$\downarrow$

calculate new ε_g with previous and present $P, T \ldots$

Boundary value problem (1)-(4):

Discretization

Solution of difference equation by Newton method

Matrix inversion (block structure)

Block elimination procedure by Henyey

Equation of State (EOS)

Consider mixture of various nuclei, atoms, (partially) ionized atoms, electrons and radiation

Ions, atoms and electrons (but: degeneracy) satisfy condition of ideal gas:

$$P_{ion} = \sum_k n_k kT$$

$$P_e = n_e kT$$

$$P_{gas} = P_{ion} + P_e$$

For practical purposes desired

$$P_{gas} \overset{!}{=} \frac{\mathcal{R}}{\mu} \rho T \quad \mu\text{: mean molecular weight}$$

$$\rho = \sum_k n_k \mu_k m_p + n_e m_e$$

$$P_{gas} = kT \left(\sum_k n_k + n_e \right)$$

$$= \frac{kT}{\mu} \left(\sum n_k \mu_k + n_e m_e / m_p \right)$$

$$\mu = \frac{\sum_k n_k \mu_k + (n_e m_e / m_p)}{\sum_k n_k + n_e}$$

μ depends on degree of ionization

Complete Ionization (stellar core)

$$n_e = \sum_k n_k Z_k$$

$$X_k = \frac{n_k \mu_k m_p}{\rho}$$

$$\sum_k X_k = 1 \; ; \quad m_e/m_p \ll 1$$

$$\mu = \frac{\sum_k \frac{n_k \mu_k m_p}{\rho}}{\sum_k (1+Z_k)\frac{1}{\mu_k} \frac{n_k \mu_k m_p}{\rho}}$$

$$\mu = \frac{1}{\sum_k \frac{1+Z_k}{\mu_k} X_k}$$

"Heavy elements": $Z_k \approx \mu_k/2 \gg 1$

$$\mu = \frac{1}{2X + \frac{3}{4}Y + \frac{1}{2}Z}$$

Partial ionization (stellar envelope)

Determine number density of atoms in their various states, various ions correspondingly and n_e according to thermodynamic equilibrium

Consider atom / ion in state i:
n_i : number density ; g_i : statistical weights

$$\frac{n_j}{n_i} = \frac{g_j}{g_i} \exp\left(-(E_j - E_i)/kT\right)$$

Boltzmann - formula

n_r : number density of r times ionized ion in ground state

$$\frac{n_{r+1}\, n_e}{n_r} = \frac{g_{r+1}}{g_r}\, 2\, \frac{(2\pi m_e kT)^{3/2}}{h^3} \exp(-\chi_r/kT)$$

χ_r : ionization energy

Saha - equation (law of mass action)

$$\left.\begin{array}{l}\text{Add up all contributions} \\ \text{charge neutrality} \\ \text{mass conservation}\end{array}\right\} \Rightarrow \mu(\rho, T) \text{ or } \mu(p, T)$$

Radiation pressure

$$P_{rad} = \frac{a}{3} T^4$$

Total pressure

$$p = p_{gas} + p_{rad} = \frac{R}{\mu} \rho T + \frac{a}{3} T^4$$

Specific internal energy:

$$u = \frac{3}{2}\frac{\mathcal{R}}{\mu}T + \frac{aT^4}{\rho} + \text{constant}$$

c_p, δ, ∇_{ad} deduced directly or by means of standard thermodynamic relations

Electron pressure at low temperature

Electrons obey Fermi-Dirac-statistics

Probability to find electron with momentum p in interval dp

$$dN_p = \frac{gV\,p^2 dp}{2\pi\hbar^2\left[\exp((\varepsilon-\mu)/kT) + 1\right]}$$

$g = 2$ statistical weight

V : volume

μ : chemical potential

$\varepsilon(p)$: energy

Number density

$$\frac{N_e}{V} = \int_0^\infty \frac{dN_p}{V}$$

Energy density

$$\frac{E}{V} = \int_0^\infty \varepsilon(p)\,\frac{dN_p}{V}$$

Thermodynamic potential Ω
(see Landau-Lifschitz)

$$\Omega = - \int_0^\infty kT \, \frac{g p^2 dp}{2\pi^2 \hbar^3} \log\left[\exp\left\{(\mu-\varepsilon)/kT\right\}+1\right]$$

General property:

$$\Omega = -pV = -p_e V$$

For Fermi-Dirac-statistics:

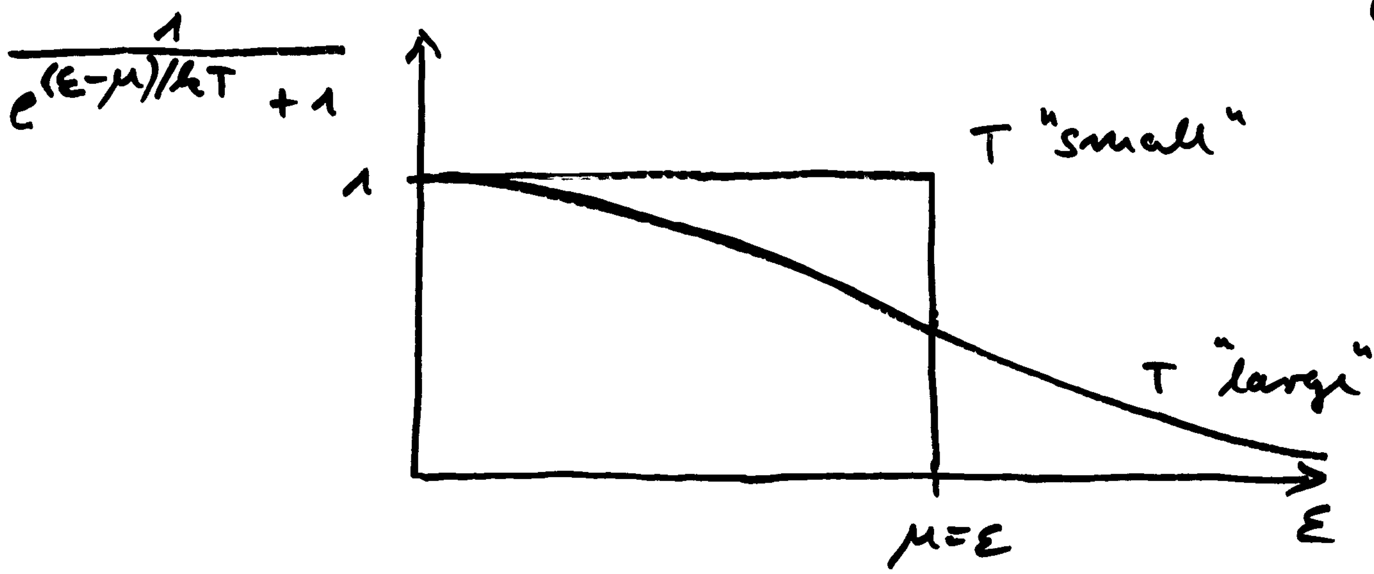
$$\left.\frac{\partial\Omega}{\partial\mu}\right|_T = -N_e$$

Non-relativistic particles

$$\varepsilon = \frac{p^2}{2m} \; ; \; m = m_e \qquad : \quad p_e V = \frac{2}{3} E$$

Relativistic particles:

$$\varepsilon = cp \qquad : \quad p_e V = \frac{1}{3} E$$



T "small": degeneracy

Complete degeneracy:

$$\frac{N_e}{V} \sim \int_0^{p_0} p^2\,dp \sim p_0^3 \quad ; \; p_0 : \varepsilon = \mu$$

$$\frac{E}{V} \sim \int_0^{p_0} \varepsilon\, p^2\,dp \;\begin{cases} \nearrow \displaystyle\int_0^{p_0} p^4\,dp \sim p_0^5 & \text{non-relativistic} \\ \searrow \displaystyle\int_0^{p_0} p^3\,dp \sim p_0^4 & \text{relativistic} \end{cases}$$

$$P_e \sim \frac{E}{V} \;\begin{cases} \nearrow p_0^5 \sim \left(\frac{N_e}{V}\right)^{5/3} \sim \rho^{5/3} & \text{non-relativistic} \\ \searrow p_0^4 \sim \left(\frac{N_e}{V}\right)^{4/3} \sim \rho^{4/3} & \text{relativistic} \end{cases}$$

"T small", condition for degeneracy

$$kT \ll \varepsilon_0$$

$\varepsilon_0 \hat{=} p_0$ maximum momentum or energy

Non-relativistic particles:

$$\varepsilon_0 \sim p_0^2 \sim \left(\frac{N_e}{V}\right)^{2/3}$$

Condition for degeneracy:

$$kT \ll \frac{\hbar^2}{m}\left(\frac{N_e}{V}\right)^{2/3}$$

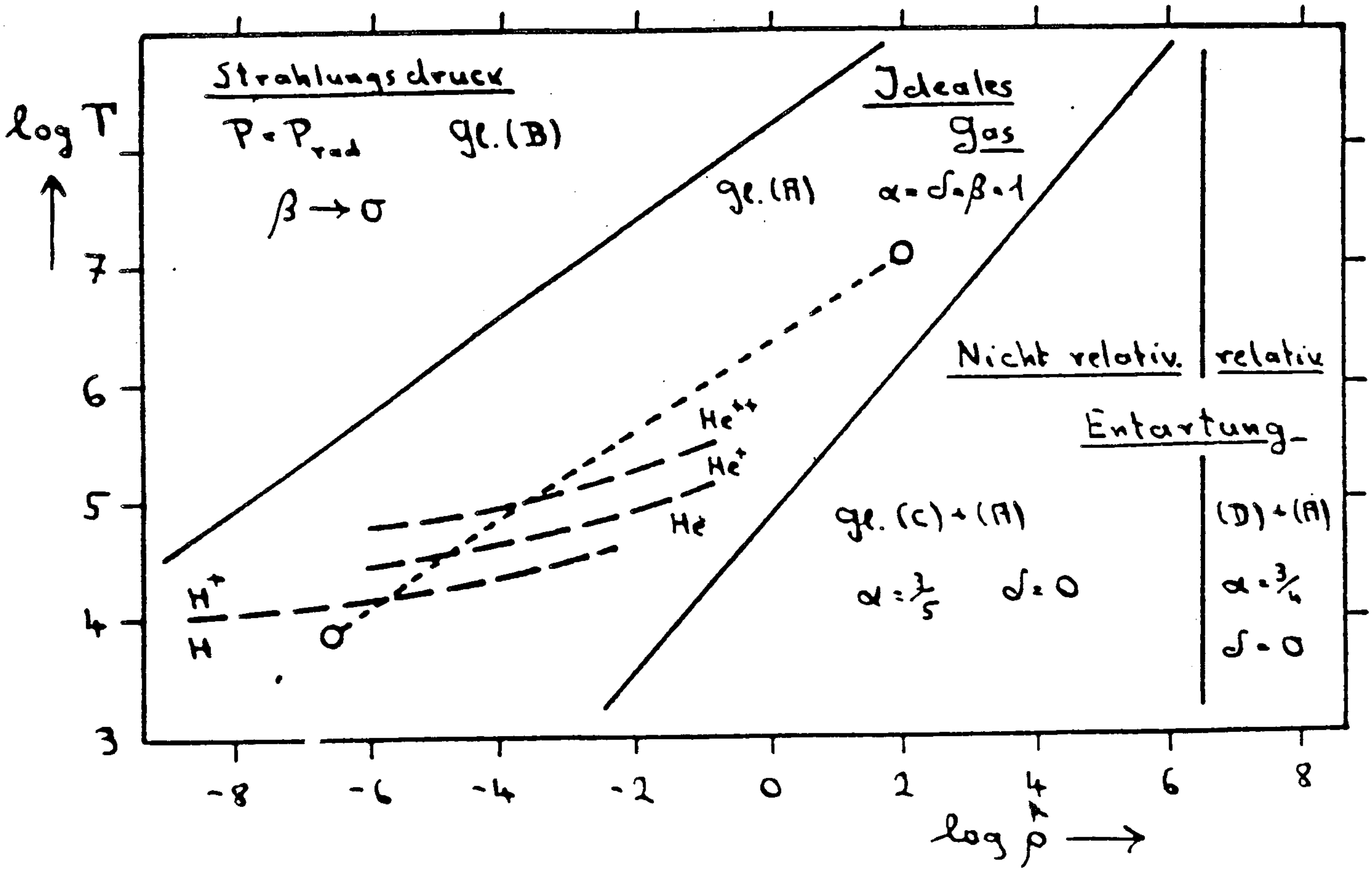
Valid for all Fermions: nuclei degenerate at lower temperatures than electrons $\rightarrow e^-$ dominate

Degeneracy:

EOS independent of T

$$P_e \sim \rho^{5/3} \quad \text{non-relativistic } e^-$$

$$P_e \sim \rho^{4/3} \quad \text{relativistic } e^-$$


log T
log ρ
Strahlungsdruck
$P = P_{rad}$
gl. (B)
$\beta \to 0$
Ideales Gas
gl. (A)
$\alpha = \delta = \beta = 1$
Nicht relativ.
relativ
Entartung
gl. (C) + (A)
$\alpha = 3/5$
$\delta = 0$
(D) + (A)
$\alpha = 3/4$
$\delta = 0$
He^{++}
He^{+}
He
H^{+}
H
3
4
5
6
7
-8
-6
-4
-2
0
2
4
6
8

Opacity

Sources

- Electron scattering (Thomson scattering)

  cross section $\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2$

  constant, frequency independent

  $$\kappa_e = \sigma_e \frac{n_e}{\rho} = 0.2 \ \frac{\mathrm{cm}^2}{\mathrm{g}} (1+X)$$

  (fully ionized matter)

- Free - free transitions

- Photo-ionisation

  Bound - free transitions

- Bound - bound transitions

- H^- opacity for $T < 6000\,K$

  $$H + e^- \rightleftarrows H^-$$

- Conduction if electrons are degenerate

Treatment of Opacity

For analytical estimates: Kramer's opacity

Power law fit:

$$\kappa(\rho, T) \sim \rho^{\alpha} T^{\beta}$$

with, e.g., $\alpha = 1$ and $\beta = -3.5$

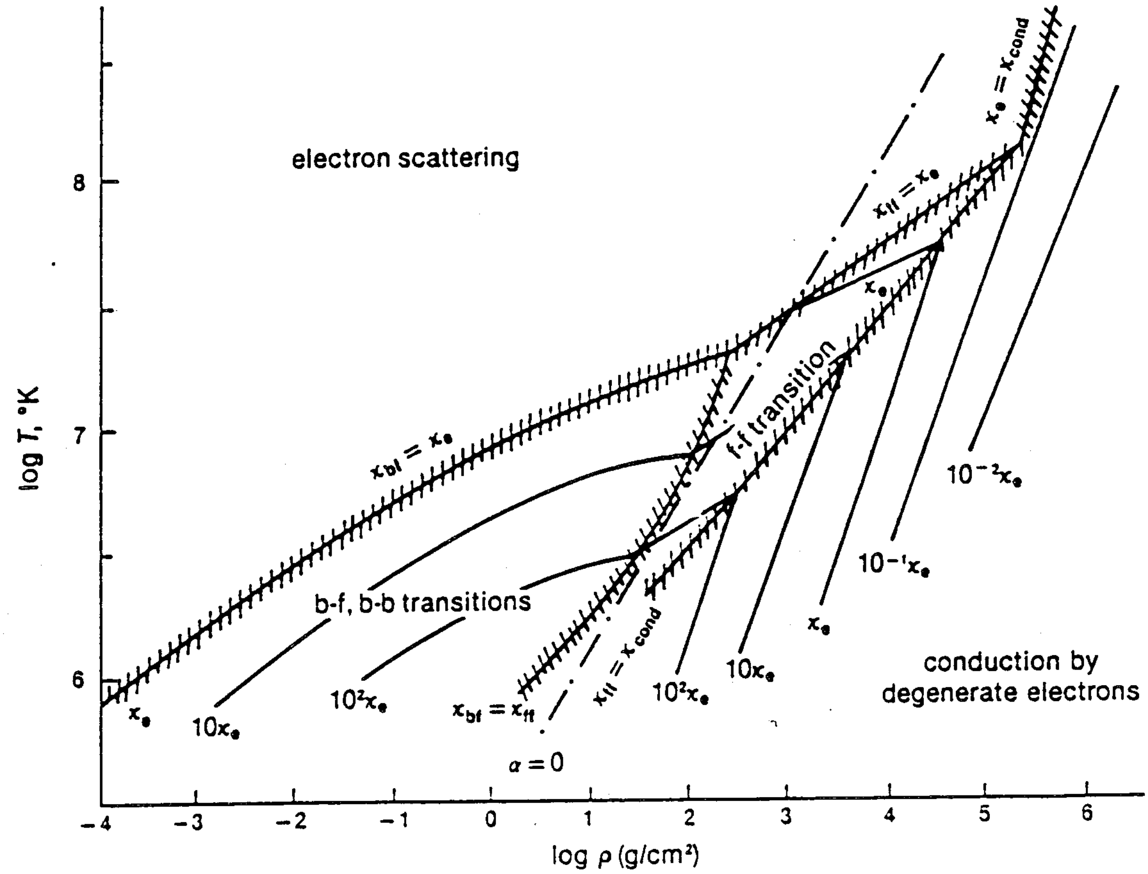
Opacity strongly dependent on chemical composition

For realistic calculations: Opacity tables

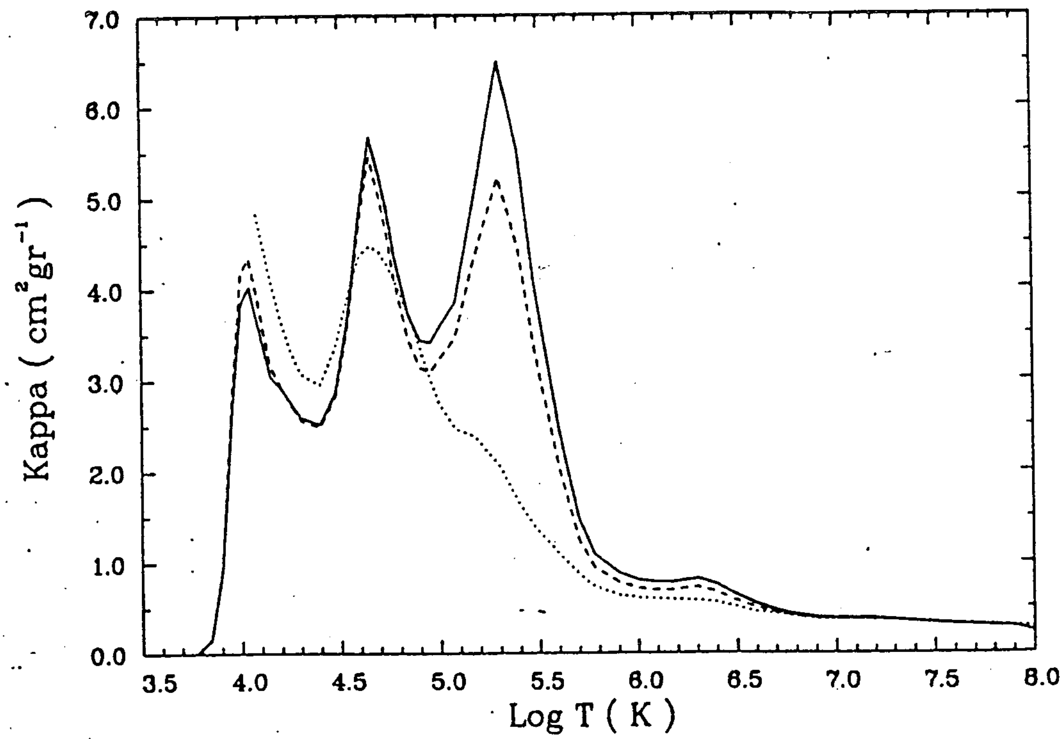
Table parameters: ρ, T, composition

Projects: LAOL : Los Alamos
OPAL : Livermore
OP : MHD

63a

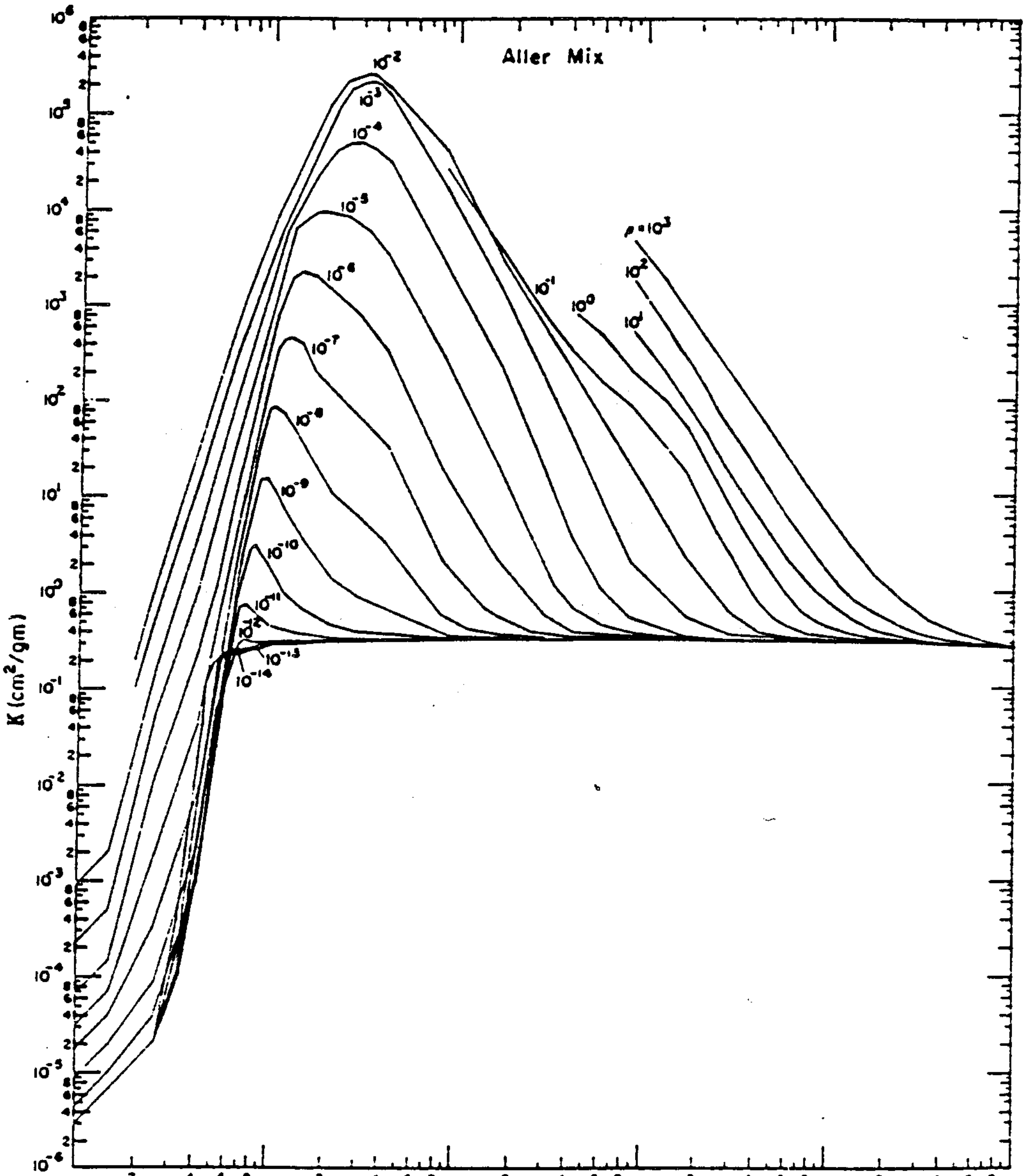


electron scattering
log T, °K
log ρ (g/cm³)
b-f, b-b transitions
f-f transition
conduction by
degenerate electrons
α = 0



Kappa (cm²gr⁻¹)
Log T (K)

63b



Nuclear Reactions
Nuclear energy generation

Consider reaction

$$A + B \longrightarrow C + D$$

Energy generation due to mass difference

$$m_A + m_B \neq m_C + m_D$$

Energy release: $\Delta Q = \Delta m c^2$

Binding energy:

$$BE := \left[Z m_p + (A-Z) m_n - m(A,Z) \right] c^2$$

energy released by forming nucleus (A,Z)

Binding energy per nucleon

$$BE/A := f$$

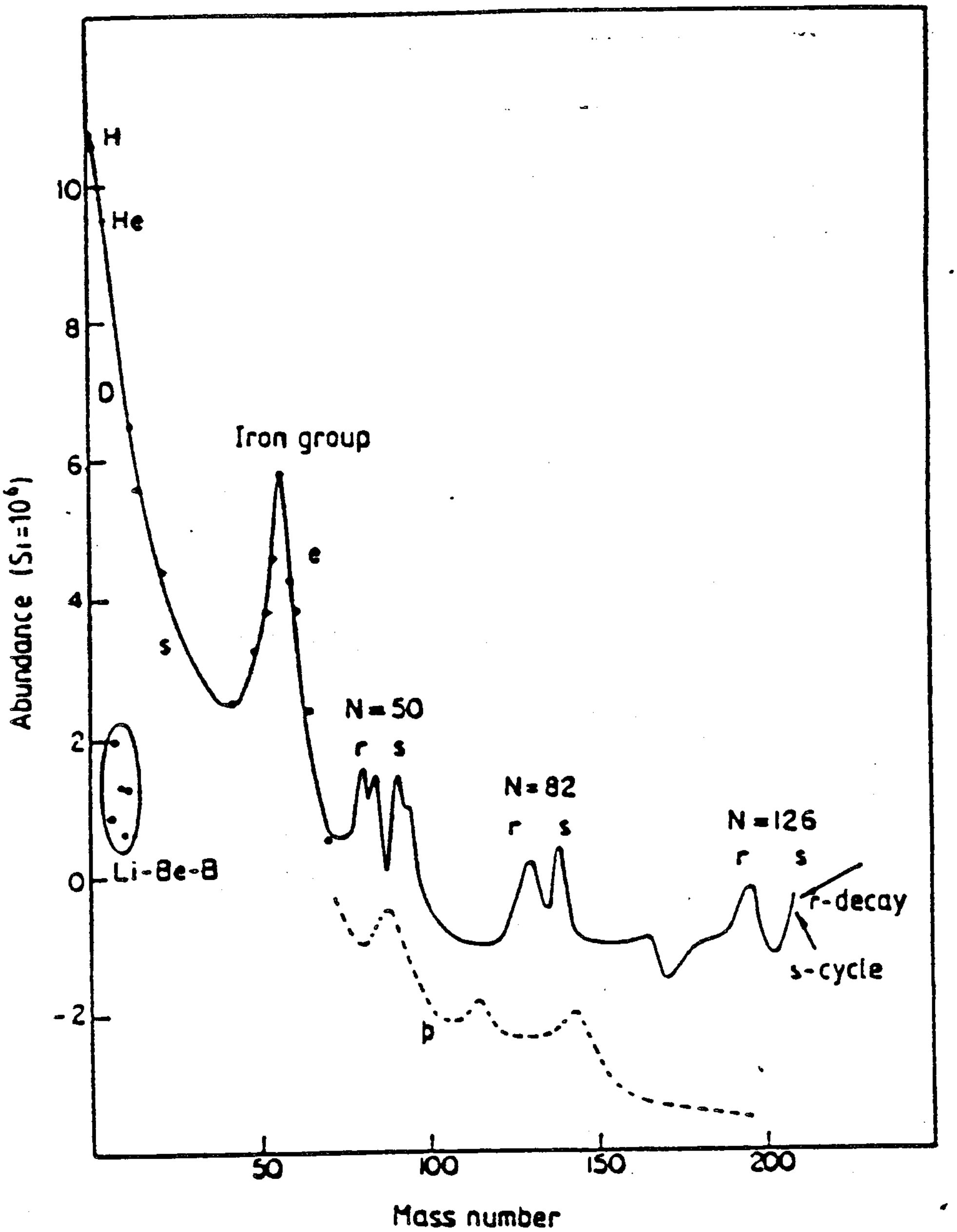
ΔQ corresponds to difference in binding energies

$$\Delta Q = BE_C + BE_D - BE_A - BE_B$$

$$= A_C f(A_C) + A_D f(A_D) - A_A f(A_A) - A_B f(A_B)$$

$\Delta Q > 0$ only for $A_A + A_B \lesssim 56$ (Fe)

$\Delta Q < 0$ otherwise



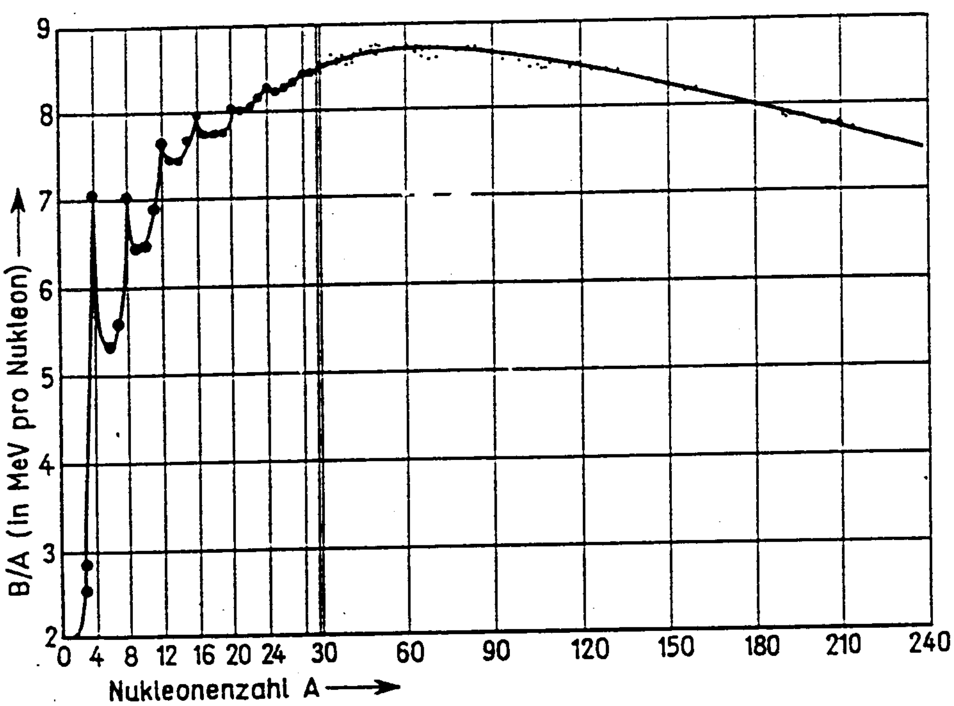
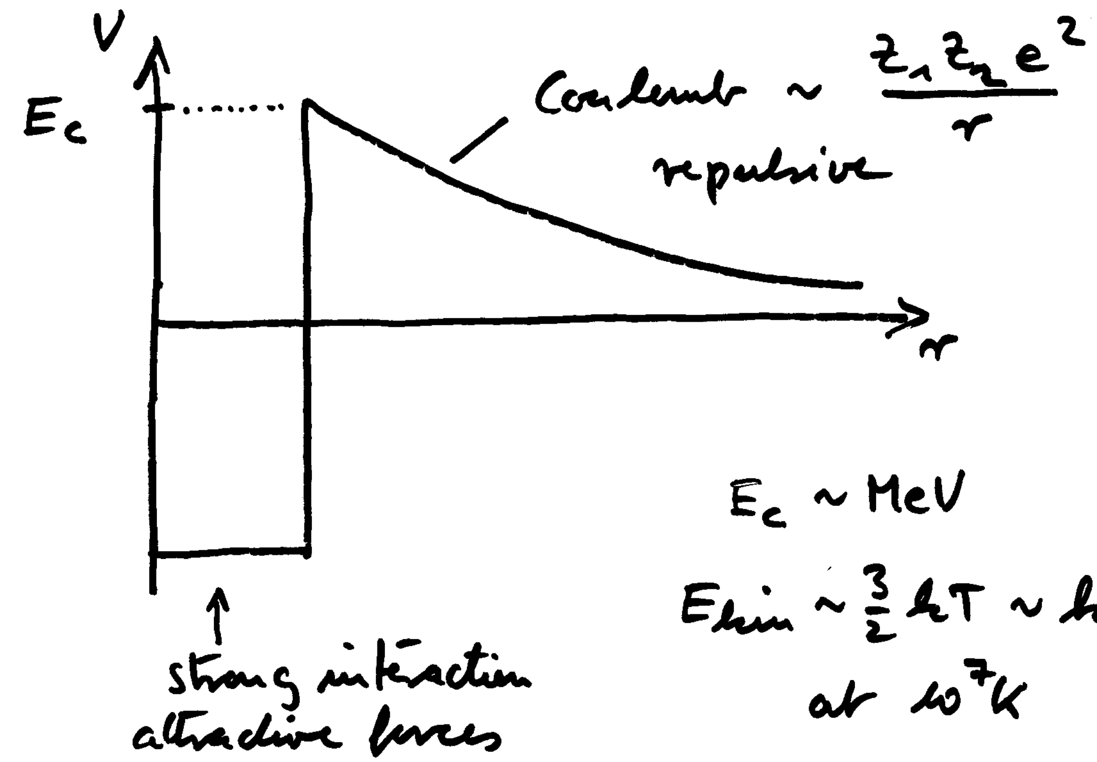


Fig.10 Bindungsenergie pro Nukleon als Funktion von A für stabile Kerne; nach [Eva 55]. Abszisse bis A = 30 gespreizt

Nuclear Potential



$E_c \sim \text{MeV}$

$E_{kin} \sim \frac{3}{2} kT \sim \text{keV}$
at 10^7 K

Nuclear Astrophysics: low energy domain

$$E_{kin} \ll E_c$$

Overcoming of Coulomb-barrier

- Maxwell distribution

$$\frac{dN}{dE} \sim e^{-E/kT}$$

particles also at high energies

- tunneling: finite penetration probability even for $E < E_c$

Reaction rate r_{12} (reactions / Volume / Time) of "particle 1" (at rest) with "particle 2"

$n_{1/2}$: number density

v : (relative) velocity

j_2 : flux of particles $j_2 = n_2 v$

$$r_{12} = n_1 j_2 \sigma_{12} = n_1 n_2 v \, \sigma_{12}(v)$$

σ_{12} : cross section

velocity distribution for particles $W(v)$

$$\int W(v)\, dv = 1$$

$$r_{12} = n_1 n_2 \int W(v) \, v \, \sigma_{12}(v) \, dv$$

$$= n_1 n_2 \langle v \sigma_{12} \rangle$$

$$\langle v \sigma_{12} \rangle \quad \Longrightarrow \quad \dot{X}_1 , \dot{X}_2 , \varepsilon_{12}$$

$$\varepsilon_{12} = \frac{\Delta Q_{12} \, r_{12}}{\rho} = \rho \frac{X_1 X_2}{m_1 m_2} \Delta Q_{12} \langle v \sigma_{12} \rangle$$

identical particles : factor 1/2

σ_{12} : if possible, experimentally

problem: low energies, extrapolation necessary

Theoretical treatment of σ

$$\sigma \sim P_R \, P_c$$

Reaction probability $P_R \sim 1/E$

Tunneling probability $P_c \sim \exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\right)$

$$\sigma(E) = \frac{S(E)}{E} \exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\right)$$

$S(E)$ constant : non-resonant reactions

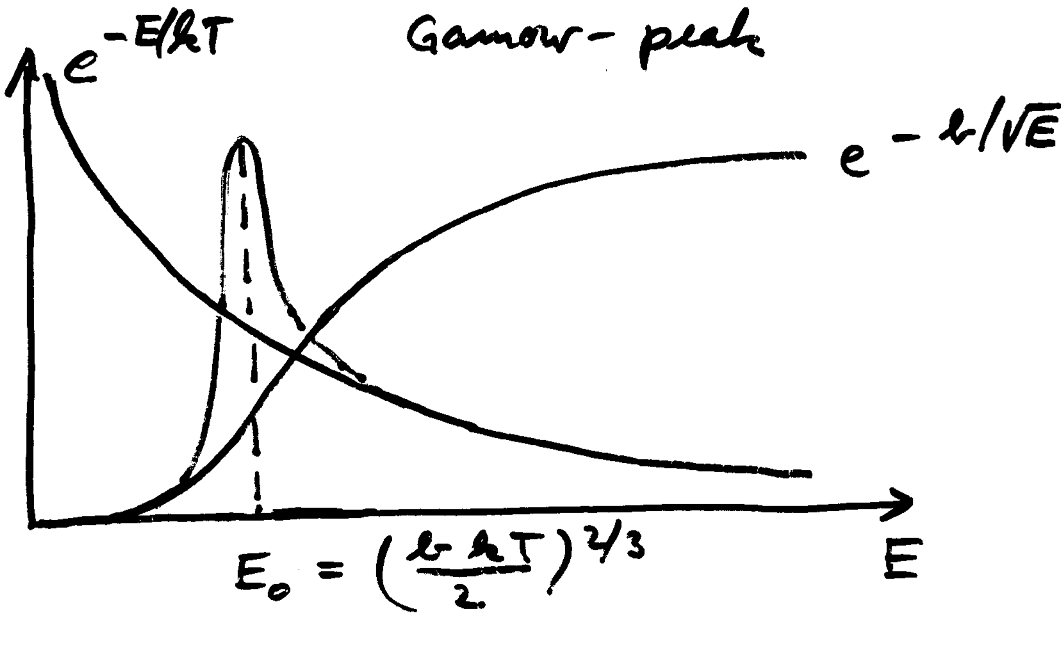
$$E = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} v^2$$

Velocity distribution : Maxwell

$$w(v)\,dv = w(E)\,dE \sim \frac{E^{1/2}}{(kT)^{3/2}} \exp\left(-E/kT\right) dE$$

$$\langle \sigma v \rangle = \int w(v)\, \sigma(v)\, v \, dv$$

$$\sim \frac{1}{(kT)^{3/2}} \int_0^\infty dE \; S(E) \exp\left(-\frac{E}{kT} - \frac{b}{\sqrt{E}}\right)$$



Approximation of Gamow-peak:

$$\exp\left(-\frac{E}{kT}-\frac{b}{\sqrt{E}}\right)\approx e^{-\tau}\exp\left[-\left(\frac{E-E_0}{\Delta/2}\right)^2\right]$$

$$\Delta=\frac{4}{\sqrt{3}}\left(E_0 kT\right)^{1/2}\quad;\tau=\frac{3E_0}{kT}\sim T^{-1/3}$$

$$\langle\sigma v\rangle\sim\frac{S(E_0)}{b}\tau^2 e^{-\tau}$$

$$\varepsilon_{12}\sim\rho\tau^2 e^{-\tau}$$

Sometimes parametrization of ε as

$$\varepsilon\sim\rho^{\lambda}T^{\nu}\qquad \nu\sim 5\ldots 40$$

Nuclear burning phases

Hydrogen Burning

$$4p \longrightarrow {}^4He + 2e^+ + 2\nu$$

$T \lesssim 1.8\,10^7\,K$: pp-chain(s)

$$\left.\begin{array}{l} {}^1H + {}^1H \longrightarrow {}^2D + e^+ + \nu \\ {}^2D + {}^1H \longrightarrow {}^3He + \gamma \\ {}^3He + {}^3He \longrightarrow {}^4He + 2\,{}^1H \end{array}\right\} \text{pp I-chain}$$

$$\left.\begin{array}{l} {}^3He + {}^4He \longrightarrow {}^7Be + \gamma \\ {}^7Be + e^- \longrightarrow {}^7Li + \nu \\ {}^7Li + {}^1H \longrightarrow 2\,{}^4He \end{array}\right\} \text{pp II-chain}$$

$$\left.\begin{array}{l} {}^7Be + {}^1H \longrightarrow {}^8B + \gamma \\ {}^8B \longrightarrow {}^8Be^* + e^+ + \nu \\ {}^8Be^* \longrightarrow 2\,{}^4He \end{array}\right\} \text{pp III-chain}$$

slowest reaction: ${}^1H(p, e^+\nu)\,{}^2D$ β decay
determines time scale and energy generation

$T \gtrsim 1.8 \cdot 10^7\,K$: CNO - cycle(s)

3 cycles, two of them:

$$
\begin{aligned}
{}^{12}C + {}^{1}H &\rightarrow {}^{13}N + \gamma \\
{}^{13}N &\rightarrow {}^{13}C + e^+ + \nu \\
{}^{13}C + {}^{1}H &\rightarrow {}^{14}N + \gamma \\
{}^{14}N + {}^{1}H &\rightarrow {}^{15}O + \gamma \\
{}^{15}O &\rightarrow {}^{15}N + e^+ + \nu \\
{}^{15}N + {}^{1}H &\rightarrow {}^{12}C + {}^{4}He
\end{aligned}
$$

Primary cycle

$$
\begin{aligned}
{}^{15}N + {}^{1}H &\rightarrow {}^{16}O + \gamma \\
{}^{16}O + {}^{1}H &\rightarrow {}^{17}F + \gamma \\
{}^{17}F &\rightarrow {}^{17}O + e^+ + \nu \\
{}^{17}O + {}^{1}H &\rightarrow {}^{14}N + {}^{4}He
\end{aligned}
$$

Secondary cycle

CNO : catalysts

slowest reaction ${}^{14}N(p,\gamma)\,{}^{15}O$
controls timescale and energy generation

Helium burning $T \sim 1..2 \; 10^8 K$
(Coulomb-barrier)

3α- reaction (Salpeter)

$$3\,{}^4He \rightarrow {}^{12}C$$

Occurs in 2 steps:

$${}^4He + {}^4He \rightleftarrows {}^8Be \qquad \tau_{{}^8Be} \sim 3 \cdot 10^{-16} sec$$

$${}^8Be + {}^4He \rightleftarrows {}^{12}C^*$$

By-reactions produce ${}^{16}O$ too

Carbon burning ashes: ${}^{20}Ne$, ${}^{23}Na$, ${}^{24}Mg$

several reactions

$${}^{12}C\left({}^{12}C, \begin{matrix}\alpha \\ p\end{matrix}\right) \begin{matrix}{}^{20}Ne \\ {}^{23}Na\left(p, \begin{matrix}\alpha \\ \gamma\end{matrix}\right) \begin{matrix}{}^{20}Ne \\ {}^{24}Mg\end{matrix}\end{matrix}$$

$$T \approx 5...7 \; 10^8 K$$

Neon burning ashes: ${}^{16}O$, ${}^{24}Mg$, ${}^{28}Si$

most important reaction:

$${}^{20}Ne\begin{pmatrix}\gamma, \alpha \\ \alpha, \gamma\end{pmatrix} \begin{matrix}{}^{16}O \\ {}^{24}Mg\,(\alpha, \gamma)\,{}^{28}Si\end{matrix}$$

$$T \approx 10^9 K$$

Oxygen burning ashes: mainly ${}^{28}Si$

very complicated reaction network

example:

$$
{}^{16}O\left({}^{16}O,\ \begin{array}{c} p \\ \alpha \\ n \end{array}\right) \begin{array}{l} {}^{31}P \\ {}^{28}Si \\ {}^{31}S\,(e^+\nu)\ {}^{31}P \end{array}
$$

$$T \approx 1.2 \ldots 1.5\ 10^9\,K$$

Silicon burning ashes: iron group nuclei e.g. ${}^{56}Fe$

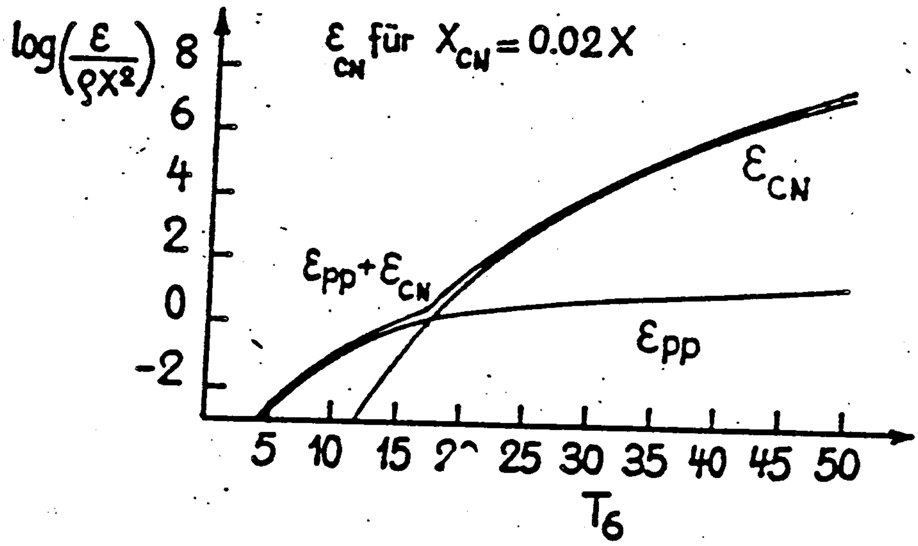
very complicated reaction network

transition to "nuclear statistical equilibrium" (NSE)

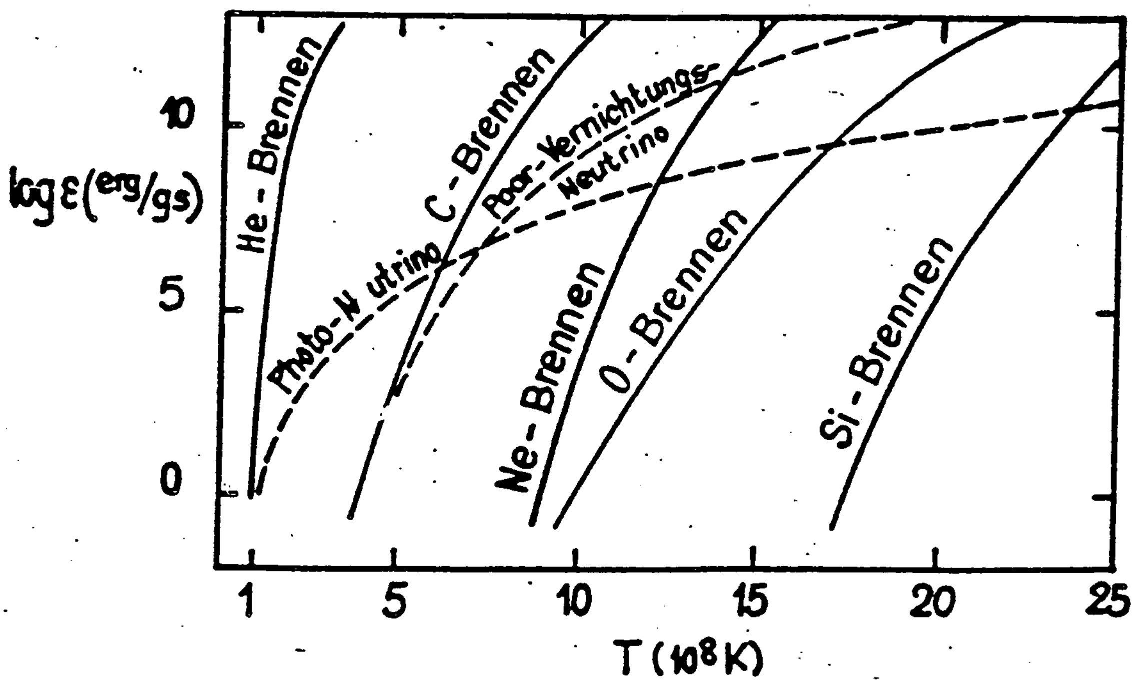
"netto" reaction:

$${}^{28}Si + {}^{28}Si \longrightarrow {}^{56}Fe$$

$$T \approx 2 \cdot 10^9\,K$$



$\log\left(\frac{\varepsilon}{\varrho X^2}\right)$
8
6
4
2
0
-2
ε_{CN} für $X_{CN} = 0.02X$
ε_{CN}
$\varepsilon_{pp}+\varepsilon_{CN}$
ε_{pp}
5 10 15 20 25 30 35 40 45 50
T_6



10
$\log\varepsilon$ (erg/gs)
5
0
He - Brennen
C - Brennen
Paar-Vernichtungs-Neutrino
Photo-Neutrino
Ne - Brennen
O - Brennen
Si - Brennen
1 5 10 15 20 25
T (10^8 K)

Polytropes

First stellar models with prescription

$$p = K \rho^{1+1/n}$$

n: polytropic index

justifications:

- equation of state $p = p(\rho)$
  e.g.: degenerate electrons
- efficient energy transport $\Rightarrow T =$ constant
  $$p = p(\rho)$$
- efficient convection : $\nabla = \nabla_{ad}$
  $$p = p(\rho, S) = p(\rho)$$

Mechanical equations

$$\frac{1}{\rho} \nabla p = - \nabla \phi \qquad \text{hydrostatic equilibrium}$$

$$\Delta \phi = 4\pi G \rho \qquad \text{Poisson}$$

System closed by polytropic relation, energy equations disregarded

Spherical symmetry

$$\frac{1}{\rho} \frac{dp}{dr} = - \frac{d\phi}{dr} \qquad (1)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho \qquad (2)$$

Insert (1) into (2), replace p by polytropic relation

$$-K\left(1+1/n\right)\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho^{1/n-1}\frac{d\rho}{dr}\right)=4\pi G\rho$$

Normalization and transformation

$$\rho=\rho_c\,\Theta^n$$

$$r=\alpha\,\xi \qquad \text{choose } \alpha=\frac{K(n+1)}{4\pi G}\rho_c^{1/n-1}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta}{d\xi}\right)=-\Theta^n \qquad \text{Lane-Emden equation}$$

$$\Theta(\xi=0)=1 \;;\; \frac{d\Theta}{d\xi}(\xi=0)=0 \quad \text{(regularity)}$$

single parameter: n

$n=0,1,5$: analytical solutions

$n<5$: $\Theta=0$ for $\xi<\infty$
first zero $\xi_1(n)$ stellar boundary

Mass:

$$M=\int_0^R\rho\,4\pi r^2dr=\alpha^3 4\pi\rho_c\underbrace{\int_0^{\xi_1}\Theta^n\xi^2d\xi}_{\mu_1(n)=-\xi_1^2\left.\frac{d\Theta}{d\xi}\right|_{\xi_1}}$$

Radius:

$$R=\alpha\,\xi_1(n)$$

Mean density:

$$\langle\rho\rangle=\frac{M}{\frac{4\pi}{3}R^3}=-3\rho_c\frac{\left.\frac{d\Theta}{d\xi}\right|_{\xi_1}}{\xi_1}$$

mean density, definition of α, mass $\Rightarrow$

Mass - Radius - Relation for Polytropes:

$$R^{3-n} = \frac{1}{4\pi}\left[\frac{K(n+1)}{G}\right]^n \left[-\xi_1^{\frac{n+1}{n-1}} \left.\frac{d\Theta}{d\xi}\right|_{\xi_1}\right]^{n-1} M^{1-n}$$

$$R^{3-n} \sim M^{1-n}$$

$n = 3$ $\quad$ M independent of R

$n = 1$ $\quad$ R " " M

$$\frac{dR}{dM} \sim \frac{1-n}{3-n} M^{-\frac{2}{3-n}}$$

$\frac{dR}{dM} > 0 \quad n < 1, \; n > 3$

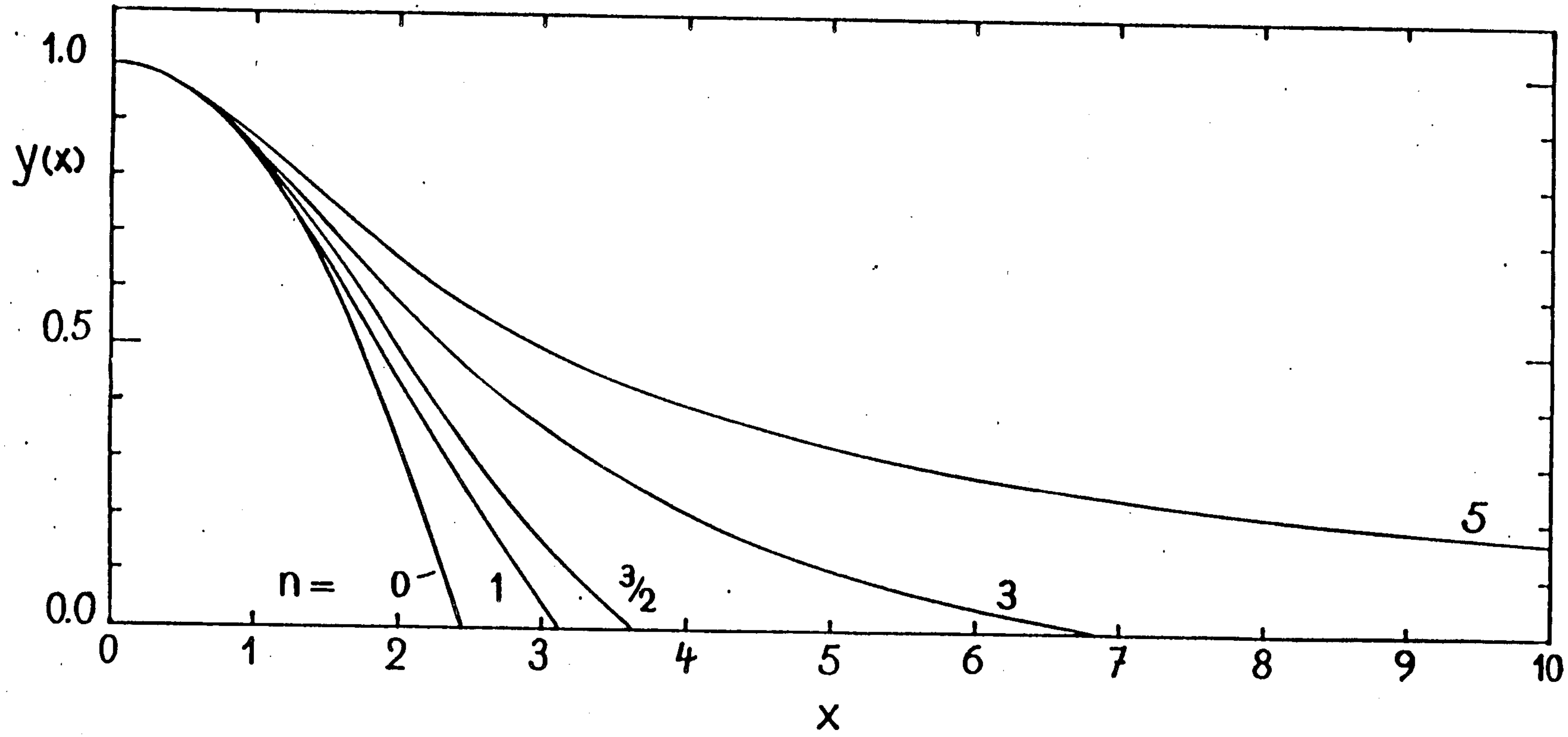
$\frac{dR}{dM} < 0 \quad 1 < n < 3$

$n = 3/2$: $\quad R \sim M^{-1/3}$

$n = 3$: $\quad$ Mass fixed by value of K

Lane - Emden - Funktionen

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -y^n$$



$x \mathrel{\hat{=}} \xi \quad y \mathrel{\hat{=}} \Theta$

Star formation

Stars form from interstellar medium (ISM)

$$\text{ISM} < \begin{cases} \text{Dust} \\ \text{gas} < \begin{cases} \text{H I} \, ; \, T \sim 100\,\text{K} \\ \text{H II} \, ; \, T \sim 10^4\,\text{K} \end{cases} \end{cases}$$

typical densities $\sim \begin{matrix} 100 \\ 0.1 \dots 1 \end{matrix}$ atoms/cm^3

Gravitational instability of interstellar cloud:

$$\left| \frac{1}{\rho} \frac{\partial p}{\partial r} \right| < \left| \frac{G M_r}{r^2} \right|$$

$$P/\rho \sim \frac{\mathcal{R}}{\mu} T < \frac{GM}{R}$$

$$M > \left(\frac{\mathcal{R} T}{\mu G} \right)^{3/2} \frac{1}{\sqrt{\rho}} := M_{\text{Jeans}}$$

Jean's criterion

$T = 100\,\text{K}$; $\rho = m_p \cdot 100/\text{cm}^3 \Rightarrow M_J \sim 3000\, M_\odot$

Isothermal collapse : $\rho \uparrow \Rightarrow M_J \downarrow$

$\Rightarrow$ smaller masses become unstable $\Rightarrow$ fragmentation

Consequence:

Stars form in clusters

Pre-main sequence evolution
Hayashi-Line

3 Phases:

1) Isothermal collaps, optically thin

2) Star becomes optically thick, T ↑
pressure stops collapse
hydrostatic core forms

3) Envelope continues to fall onto hydrostatic core: "accretion phase"

$M \lesssim 3 M_\odot$: accretion phase finished before nuclear reactions start

$M \gtrsim 3 M_\odot$: accretion continues when nuclear reactions start

Caveat: These phases of star formation still poorly understood

Characteristics at the end of phase 3:

Low temperature ($\lesssim 2000$ K) → high opacity
high densities
→ objects are fully convective
→ chemical homogeneity
Stellar evolution starts chemically homogeneous

High densities $\rightarrow$ Convection efficient

Stratification adiabatic $\nabla = \nabla_{ad}$

Properties of matter

$$p = \frac{\mathcal{R}}{\mu} \rho T \qquad c_v = \frac{3}{2} \frac{\mathcal{R}}{\mu}$$

$$T\,dS = 0 = c_v\, dT - p/\rho^2 d\rho$$

$$0 = \frac{3}{2}\left(\frac{1}{\rho} dp - p/\rho^2 d\rho\right) - p/\rho^2 d\rho$$

$$\frac{dp}{p} = 5/3 \frac{d\rho}{\rho} \qquad p = K \rho^{5/3} \quad ; \; p = K' T^{5/2}$$

$$K' \sim K^{-3/2}$$

Stellar interior described by polytrope $n = 3/2$

$$K \sim R^{\frac{3-n}{n}} M^{-\frac{1-n}{n}}$$

$$K' \sim M^{-1/2} R^{-3/2}$$

$$p = K' T^{5/2}$$

Photosphere: $$P_{eff} = \frac{2}{3} \frac{GM}{R^2} \frac{1}{\kappa(P_{eff}, T_{eff})}$$

$$L = 4\pi R^2 \sigma T_{eff}^4$$

Require $p = P_{eff}$ $\quad T = T_{eff}$

Eliminate R, $p = P_{eff} \Rightarrow T_{eff}(L, M)$

Approximation

$$\log T_{eff} \approx \frac{1}{80} \log L + \frac{1}{6} \log M + \text{constant}$$

Hayashi-Line in HRD

Meaning of the Hayashi-Line

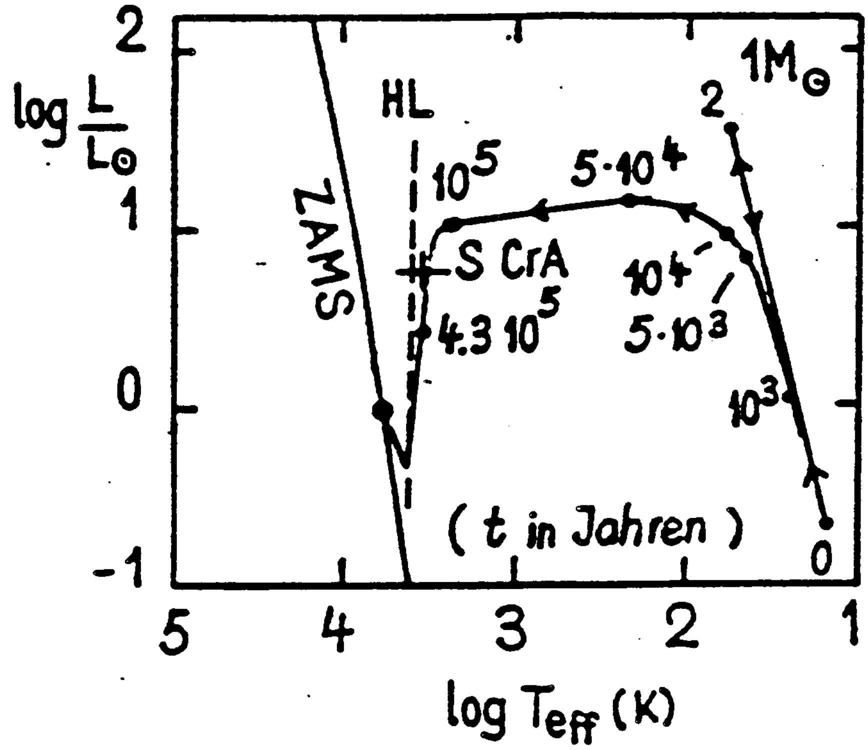
HL separates domains with $\bar{\nabla} < \bar{\nabla}_{ad}$
and $\bar{\nabla} > \bar{\nabla}_{ad}$ in the HRD by construction

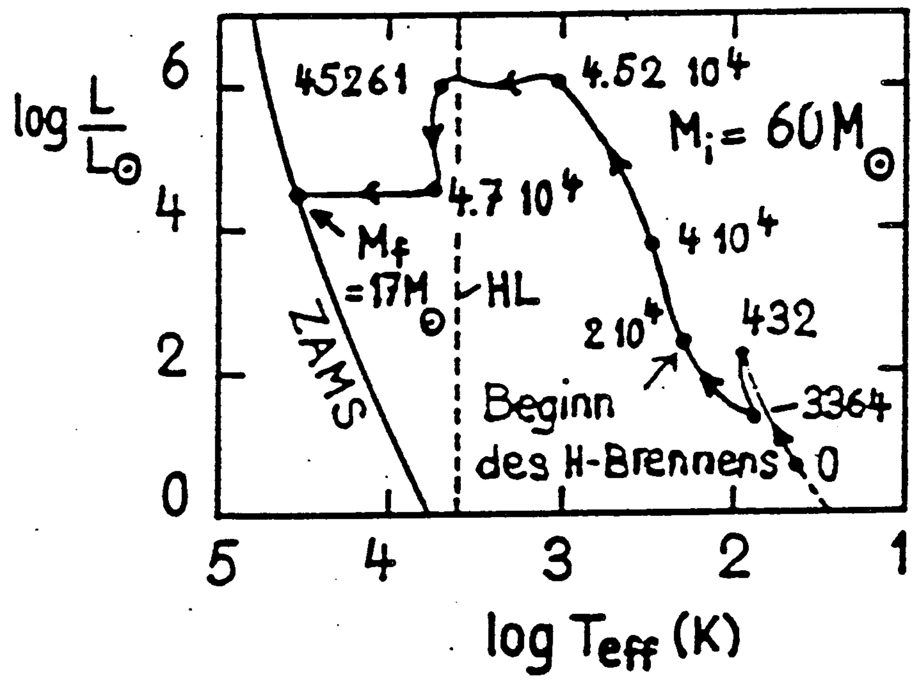
$\bar{\nabla} < \bar{\nabla}_{ad}$: star partially radiative (left to HL)

$\bar{\nabla} = \bar{\nabla}_{ad}$: HL

$\bar{\nabla} > \bar{\nabla}_{ad}$: right to HL
convection extremely strong
reduces $\bar{\nabla}$ to $\bar{\nabla}_{ad}$
reaction on dynamical timescales
No hydrostatic equilibrium
"forbidden region"

Star formation ends on HL
First hydrostatic configuration on HL
Fully convective stars lie and evolve on HL
HL is the right boundary of HRD
Region right to HL forbidden for hydrostatic stars


log L/L⊙
2
1
0
-1
5
4
3
2
1
log Teff (K)
1M⊙
HL
ZAMS
S CrA
10^5
$5\cdot10^4$
10^4
$5\cdot10^3$
10^3
$4.3\ 10^5$
2
0
(t in Jahren)

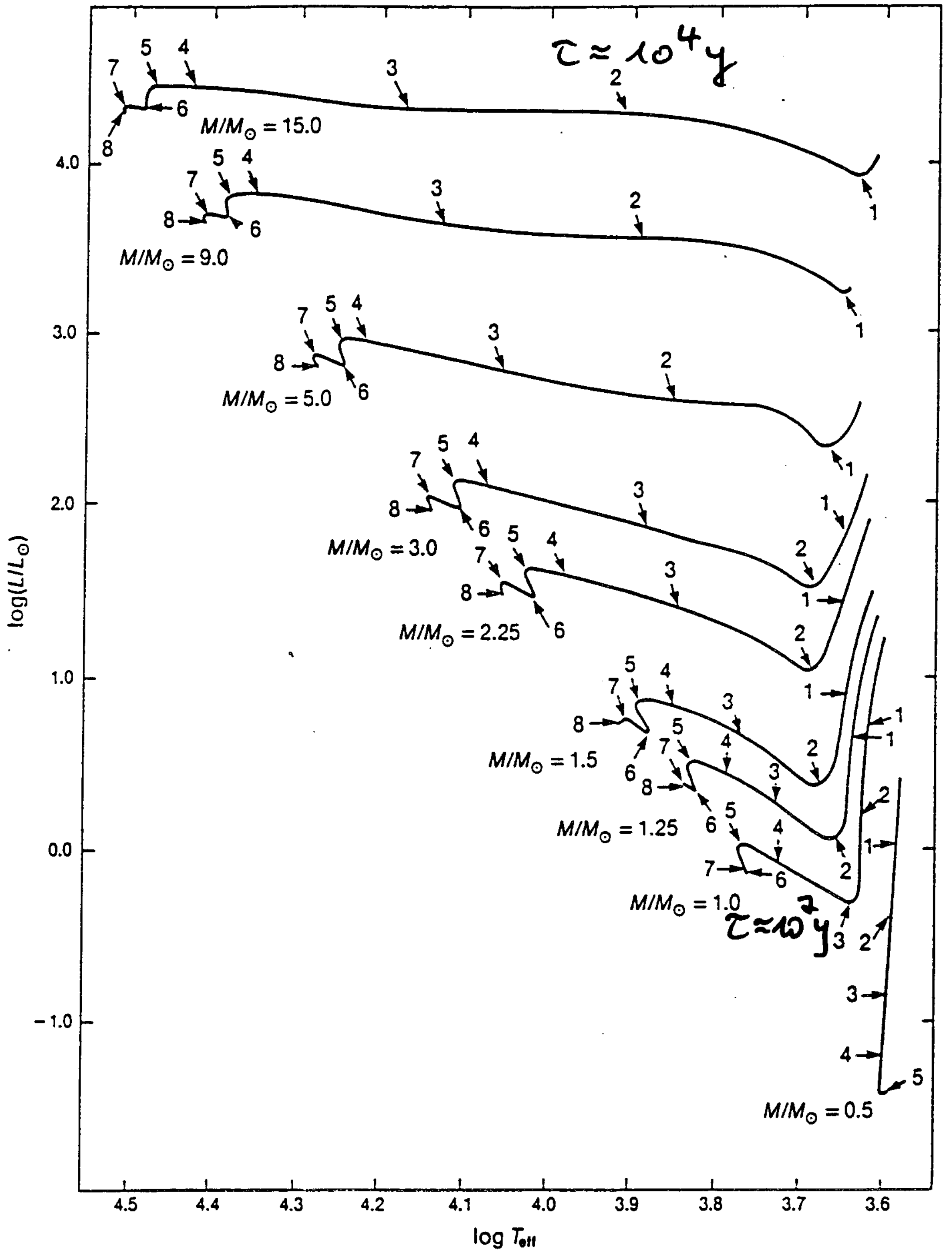

log L/L⊙
6
4
2
0
5
4
3
2
1
log Teff (K)
$M_i = 60 M_\odot$
45261
$4.52\ 10^4$
$4.7\ 10^4$
$4\ 10^4$
$2\ 10^4$
432
3364
0
$M_f = 17 M_\odot$
HL
ZAMS
Beginn
des H-Brennens

Evolution after reaching HL:
Source of energy: E_G
Timescale of evolution: τ_{KH}
Virial theorem: $T \uparrow$
} pre-main sequence evolution

Pre-main sequence phase finished if nuclear burning starts
= definition of "zero age main sequence" ZAMS

ZAMS: Chemical composition homogeneous
"Primordial" $X \sim 0.7$, $Y \sim 0.28$, $Z \sim 0.02$
$[X \sim 0.75, Y \sim 0.25, Z = 0]$

Main sequence: Location of H-burning
Phase of H-burning


τ ≈ 10⁴ y
τ ≈ 10⁷ y
$M/M_\odot = 15.0$
$M/M_\odot = 9.0$
$M/M_\odot = 5.0$
$M/M_\odot = 3.0$
$M/M_\odot = 2.25$
$M/M_\odot = 1.5$
$M/M_\odot = 1.25$
$M/M_\odot = 1.0$
$M/M_\odot = 0.5$
$\log(L/L_\odot)$
4.0
3.0
2.0
1.0
0.0
−1.0
$\log T_{\rm eff}$
4.5
4.4
4.3
4.2
4.1
4.0
3.9
3.8
3.7
3.6

Main sequence

MS: H-burning by definition

Timescale τ_{nuc}

Thermal equilibrium $\frac{dp}{dt} = 0$, $\frac{dT}{dt} = 0$

Parameters: μ, M

Position of MS: $L(T_{eff})$?

Estimates

EOS: $P = \frac{R}{\mu} \rho T$

Energy generation $\varepsilon \sim \rho T^{\nu}$

$$\left.\begin{array}{l} \dfrac{r}{M_r} \sim \dfrac{1}{r^2 \rho} \\[2ex] \dfrac{P}{M_r} \sim \dfrac{M_r}{r^4} \end{array}\right\} P/\rho \sim \frac{M_r}{r}$$

$$\left.\begin{array}{l} P/\rho \sim \dfrac{M_r}{r} \\[2ex] P/\rho \sim T/\mu \end{array}\right\} rT \sim \mu M_r$$

$$\left.\begin{array}{l} rT \sim \mu M_r \\[2ex] T/M_r \sim \dfrac{L_r \kappa}{r^4 T^3} \end{array}\right\}$$

$$L_r \sim \frac{1}{\kappa} \mu^4 M_r^3$$

Mass-luminosity-relation

Energy equation ?

Massive stars evolve faster

Energy equation

$$L_r \sim M_r \varepsilon \sim M_r \rho T^{\nu} \sim M_r \frac{M_r}{r^3}\left(\mu \frac{M_r}{r}\right)^{\nu}$$

$$L_r \sim \mu^4 M_r^3$$

Mass – Radius – relation

$$r \sim \mu^{\frac{\nu-4}{\nu+3}} M_r^{\frac{\nu-1}{\nu+3}}$$

Main sequence : $\nu \approx 13$

$$R \sim M^{3/4}$$
$$L \sim M^3 \quad ; \quad R \sim L^{1/4}$$
$$L \sim R^2 T_{eff}^4 \quad ; \quad L^{1/2} \sim T_{eff}^4$$

$$\log L \sim 8 \log T_{eff} + \text{constant}$$ function of ν

Mass varies along main sequence

compare R = constant :

$$\log L \sim 4 \log T_{eff} + \text{constant}$$

Evolution on MS during H burning

assume fixed mass, μ changes

$$L \sim \mu^4$$
$$R \sim \mu^{\frac{\nu-4}{\nu+3}}$$

Stefan - Boltzmann:

$$\log L \sim \frac{\nu+3}{\nu+10} \, 8 \log T_{eff} + \text{constant}$$

Main sequence

$$\log L \sim \quad 8 \log T_{eff} + \text{constant}$$

Evolution would proceed to the left of MS!

Numerical results: stars evolve off to the right!

Reason: Chemical inhomogeneity

Timescales

$$\frac{\tau_{H \to He}}{\tau} \approx 0.8 \dots 0.9$$

H-burning longest timescale, most stars on MS

$$\tau_{H \to He} \sim \frac{M}{L} \sim \frac{1}{M^2} \approx 6 \cdot 10^9 \, (M/M_\odot)^{-2} \text{ years}$$

Stars evolve off the MS to the right

Higher masses earlier (faster) than low masses

$\Rightarrow$ Occurrence of a "knee" in the isochrones of clusters

Internal structure on MS

2 Sections

$M \gtrsim 1.5\, M_\odot$

CNO - cycle(s)

ν large

core convective

envelope radiative

$M \lesssim 1.5\, M_\odot$

pp - chain(s)

ν small

core radiative

envelope convective

onset of convection
controlled by opacity and energy generation

End of Main sequence evolution: hydrogen exhaustion in the stellar core

Post-Main sequence evolution

Scheme

$$\begin{tikzcd}
\text{nuclear fuel exhausted} \arrow[r] & \text{contraction } E_G \arrow[d] \\
\text{ignition of next burning} \arrow[u] & \text{temperature increase virial theorem} \arrow[l]
\end{tikzcd}$$

Structure of burning: Core

Shell

Result:
Generation of onion-skin structure

Whether a burning phase is reached sensitively depends on mass

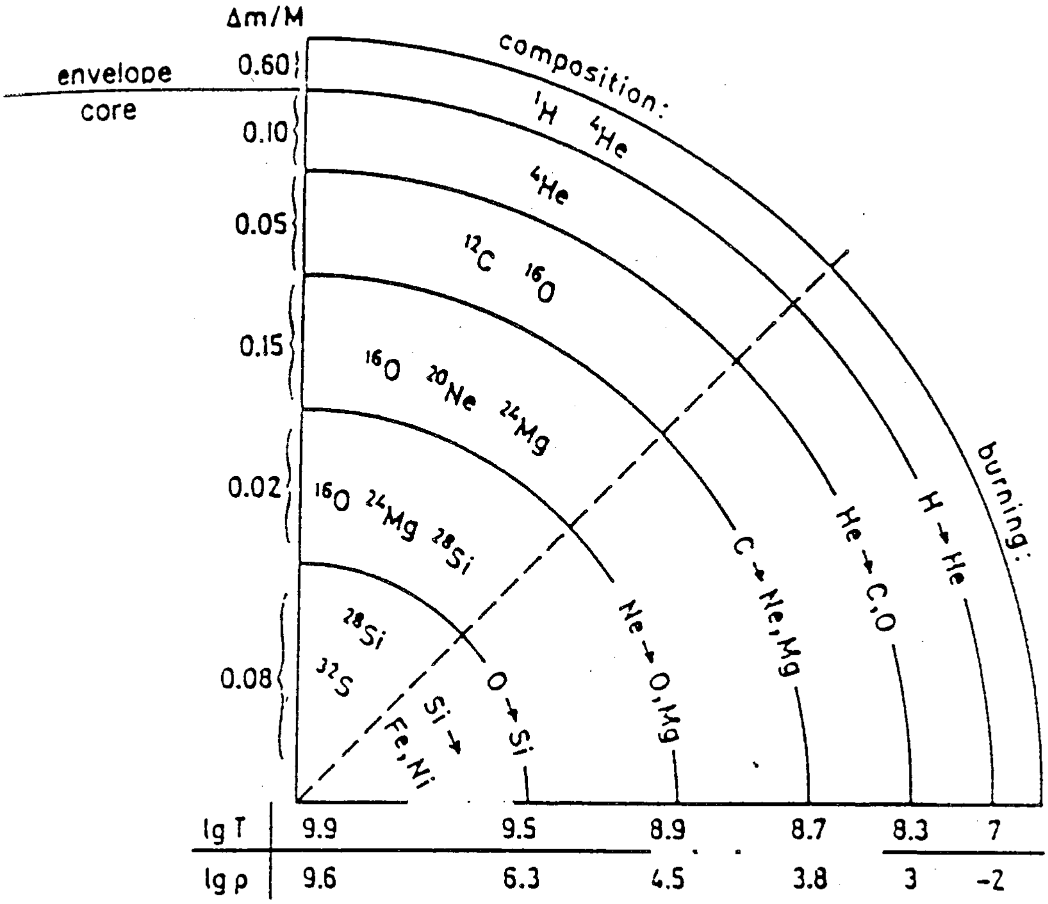


Fig. 5.8 Schematic illustration (not to scale) of the 'onion-skin' structure' in the interior of a highly evolved massive star ($25M_\odot$). Numbers along the vertical axis show some typical values of the mass fraction, while those along the horizontal axis indicate temperatures and densities (gm cm^{-3}). Adapted from R. Kippenhahn & A. Weigert, *Stellar Structure and Evolution*, Springer-Verlag 1990.

Consider homologous contraction
with time

homologous: relative structure conserved

$$\left.\begin{array}{l} \dfrac{\dot\rho}{\rho} = -3\,\dfrac{\dot r}{r} \\ \text{mass conservation} \\ \dfrac{\dot P}{P} = -4\,\dfrac{\dot r}{r} \\ \text{hydrostatic equilibrium} \\ \dfrac{\dot\rho}{\rho} = \alpha\,\dfrac{\dot P}{P} - \delta\,\dfrac{\dot T}{T} \\ \text{equation of state} \end{array}\right\} \quad \frac{\dot T}{T} = \frac{4\alpha-3}{3\delta}\,\frac{\dot\rho}{\rho}$$

ideal gas : $\alpha = \delta = 1$

$$T \sim \rho^{1/3}$$

constant degree of degeneracy:

$$T \sim \rho^{2/3}$$

$\Rightarrow$ degree of degeneracy increases during contraction

complete nonrelativistic degeneracy:

temperature independence : $\delta = 0$

pressure-density relation : $\alpha = 3/5$

During contraction δ varies from 1 to 0
α varies from 1 to 3/5

$$\frac{\dot{T}}{T} > 0 \quad \text{for } \alpha > 3/4$$

$$\frac{\dot{T}}{T} < 0 \quad \text{for } \alpha < 3/4$$

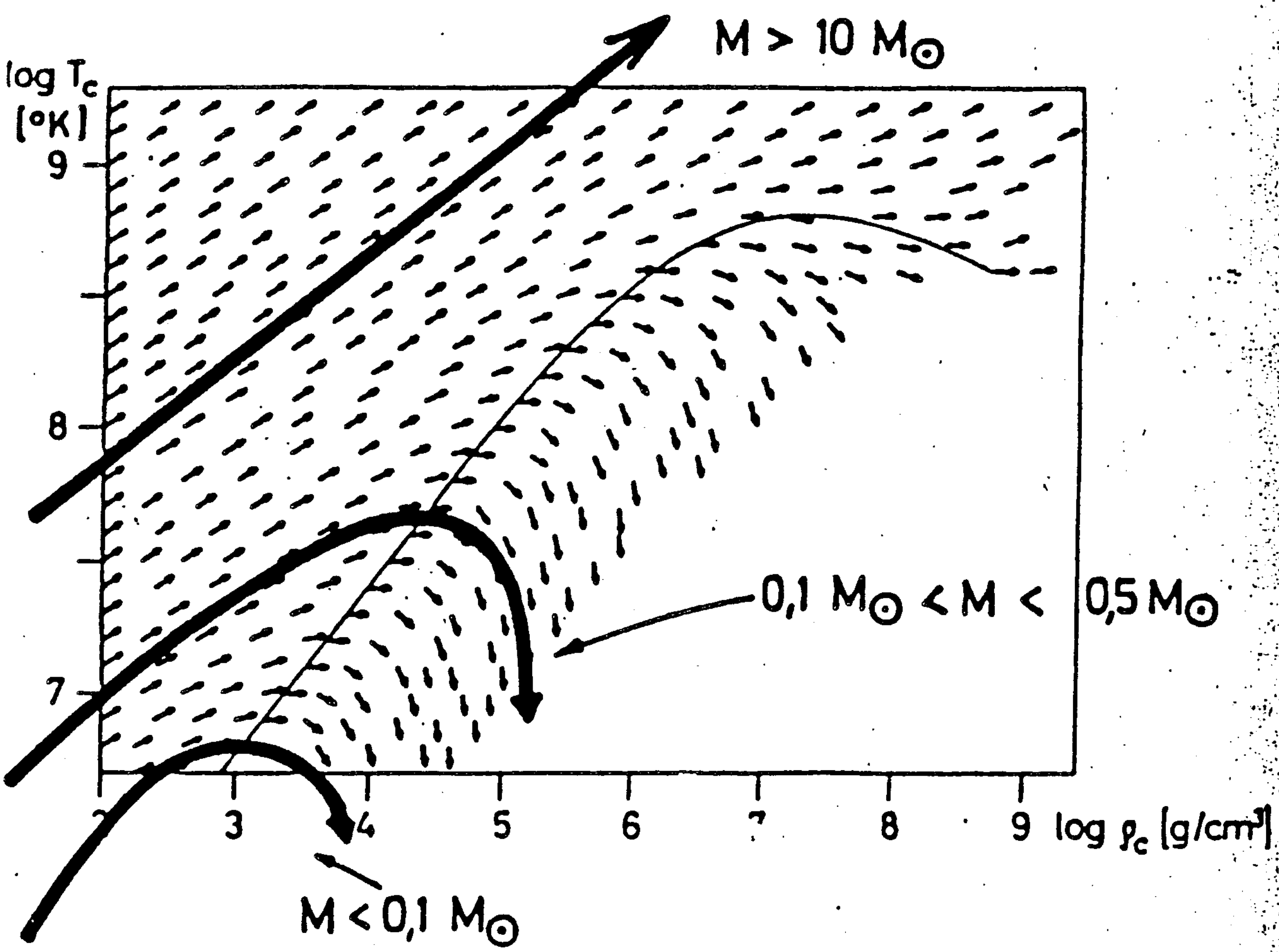
Degeneracy: T decreases for contraction

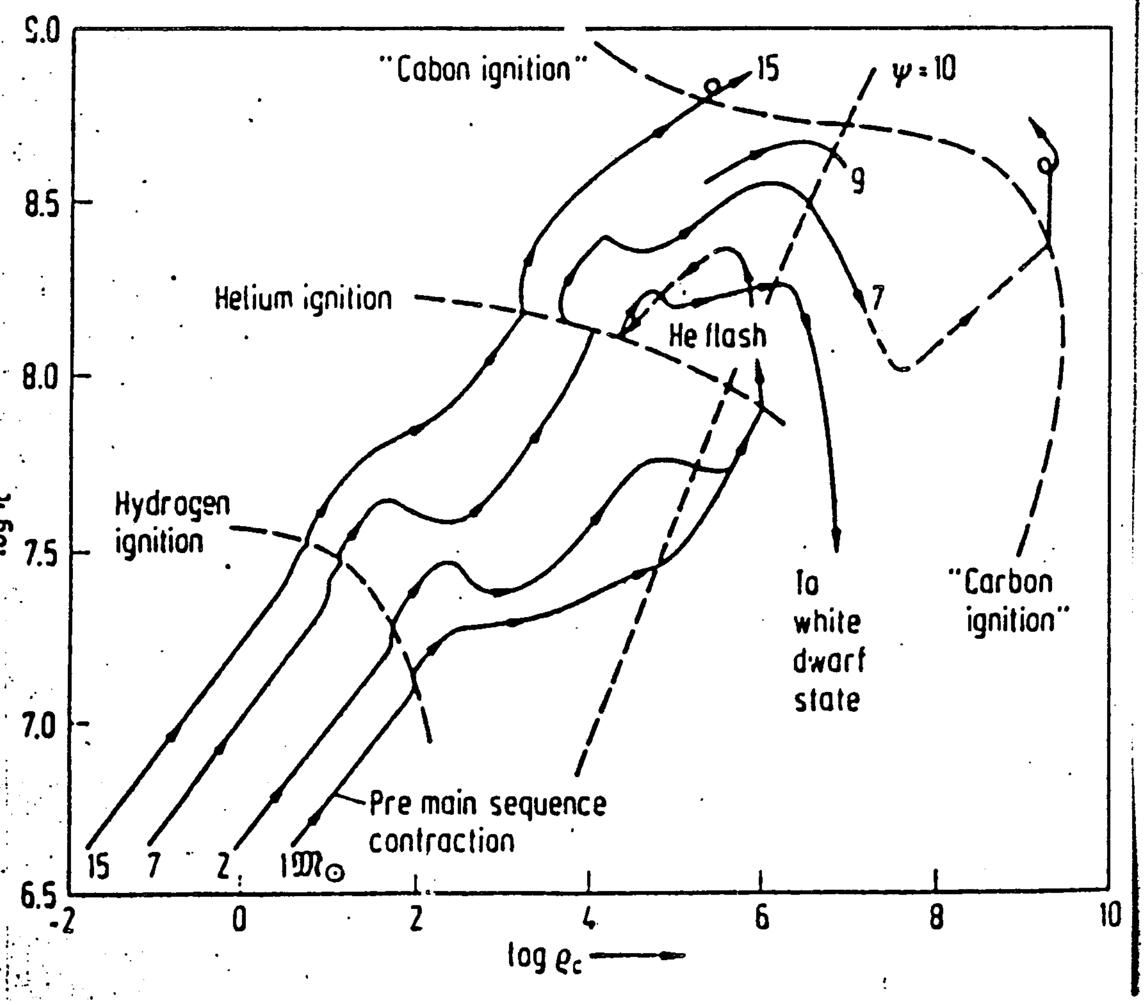
Consequence: higher burning phases will not be reached if degeneracy sets in

$\exists$ maximum temperature T during transition and contraction from ideal nondegenerate matter to degenerate electron gas

At which stage degeneracy is reached, depends on mass

Virial theorem ?


M > 10 M☉
log T_c [°K]
9
8
7
3
4
5
6
7
8
9
log ρ_c [g/cm³]
0,1 M☉ < M < 0,5 M☉
M < 0,1 M☉


"Cabon ignition"
15
ψ = 10
9
7
Helium ignition
He flash
Hydrogen ignition
To white dwarf state
"Carbon ignition"
Pre main sequence contraction
15
7
2
1𝔐☉
log T_c
log ρ_c
9.0
8.5
8.0
7.5
7.0
6.5
-2
0
2
4
6
8
10

Degeneracy before H-burning : $M < 0.08\, M_\odot$
" " He " : $M_{He} < 0.35\, M_\odot$
" " C " : $M_C < 0.9\, M_\odot$

Violation of homology: shell burning

Temperature increase in degenerate regime

$\Rightarrow$ Ignition of burning under degenerate conditions

He - flash   C - flash

Burning under degenerate conditions unstable

Degeneracy : $p = p(\rho)$

Nuclear energy release

$$dQ = c_v\, dT$$

Extreme rise of temperature

$$\varepsilon \sim T^\nu$$

Overproduction of nuclear energy

$\Rightarrow$ Instability, "flash"

Final product of stellar evolution

Degenerate He cores: $M_{initial}/M_\odot \lesssim 2.5$

" C/O " : $2.5 \lesssim M_{ini}/M_\odot \lesssim 8$

Evolutionary tracks
Interior Structure

Representative models : $M = 5 M_\odot$ and $M = 7 M_\odot$

Characteristics

Evolution from main sequence to red giant

Timescales, Hertzsprung - gap

Several "loops", crossing of

Cepheid strip

Internal structure, core - shell burning

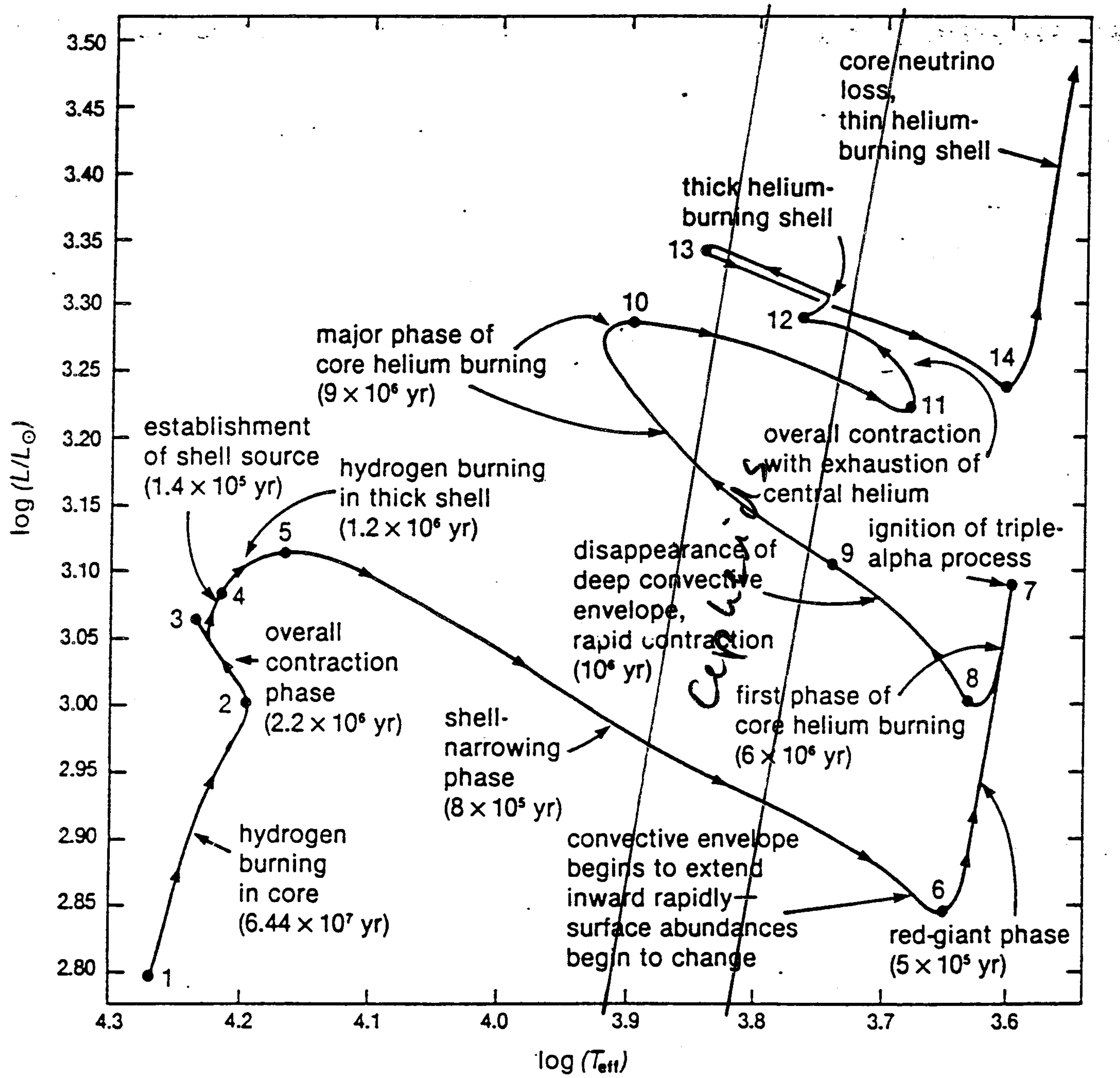
convection zones

Mirror principle at shell burning

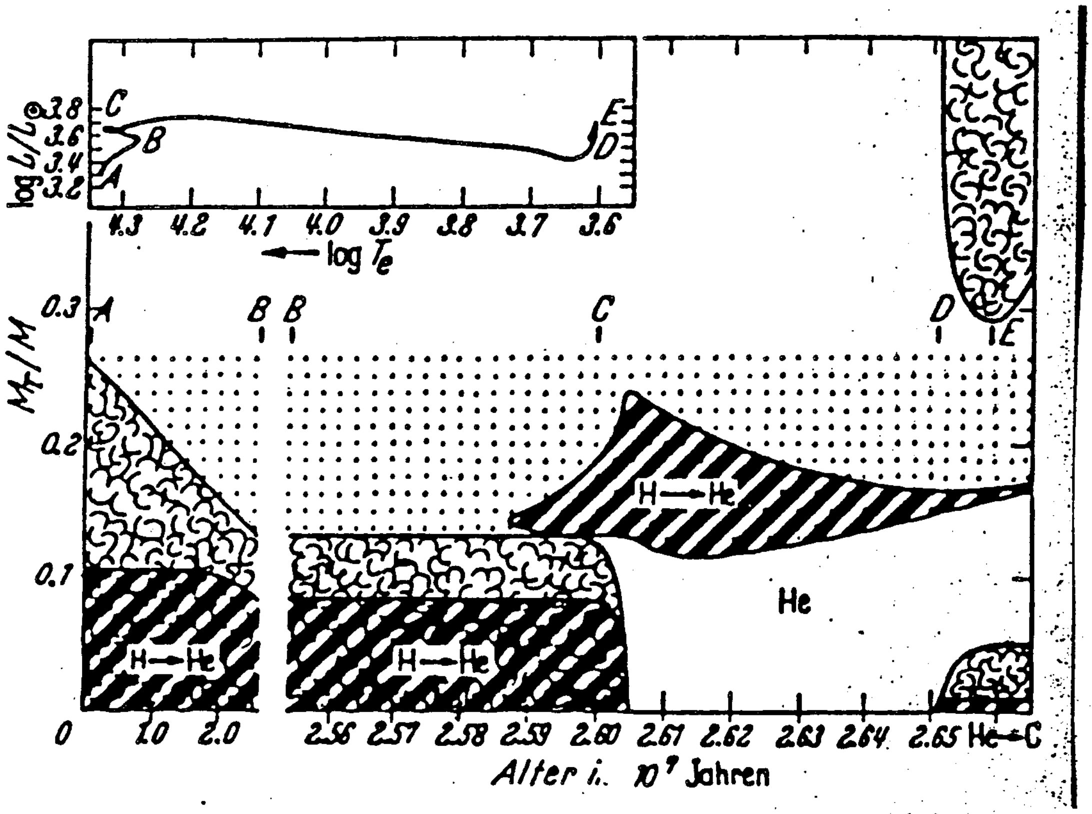
core contraction - envelope expansion

Why do stars become red giants?

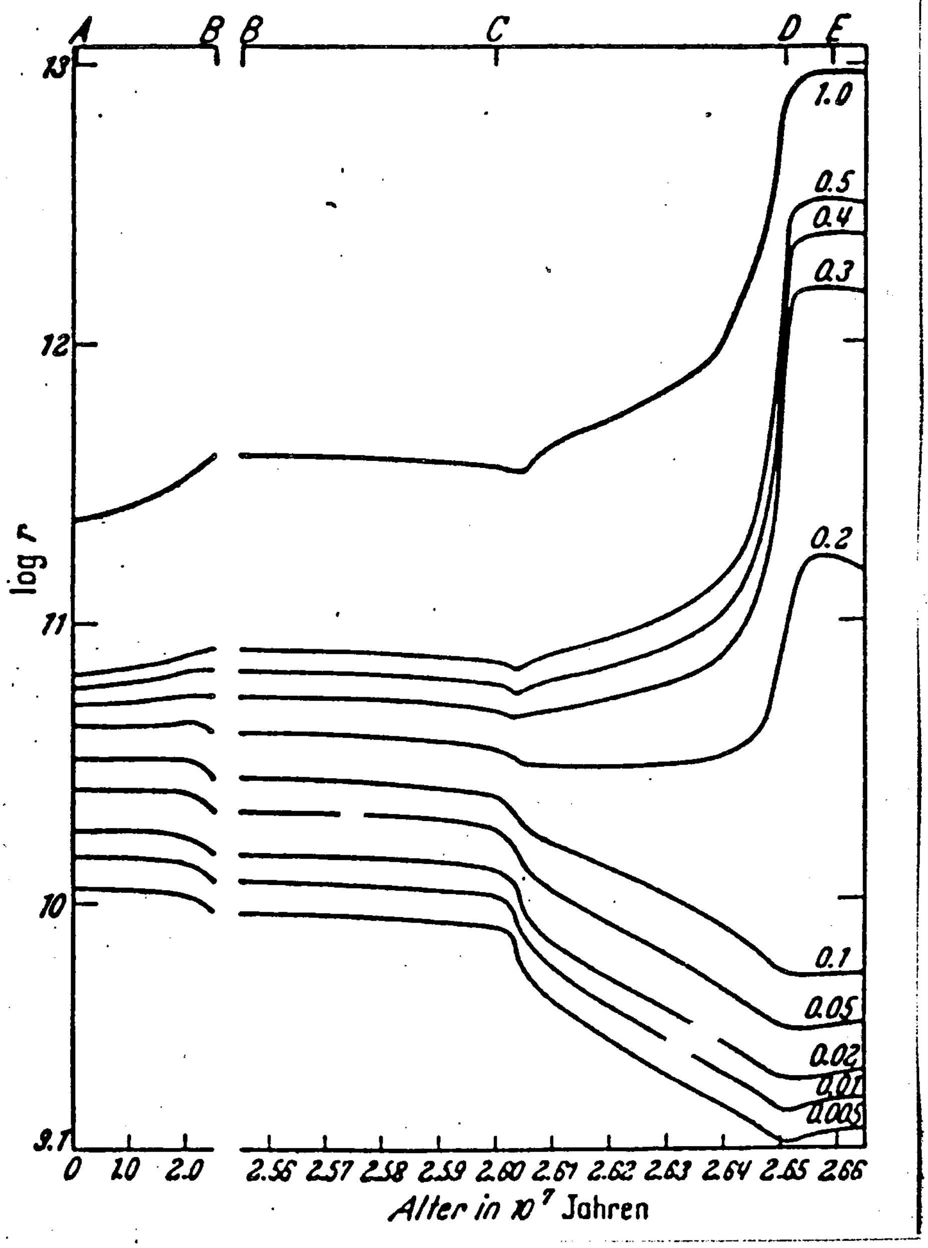
$M = 5M_\odot$


log $(L/L_\odot)$
log (T_{eff})
3.50
3.45
3.40
3.35
3.30
3.25
3.20
3.15
3.10
3.05
3.00
2.95
2.90
2.85
2.80
4.3
4.2
4.1
4.0
3.9
3.8
3.7
3.6
1
2
3
4
5
6
7
8
9
10
11
12
13
14
hydrogen burning in core (6.44×10^7 yr)
overall contraction phase (2.2×10^6 yr)
establishment of shell source (1.4×10^5 yr)
hydrogen burning in thick shell (1.2×10^6 yr)
shell-narrowing phase (8×10^5 yr)
convective envelope begins to extend inward rapidly— surface abundances begin to change
red-giant phase (5×10^5 yr)
ignition of triple-alpha process
first phase of core helium burning (6×10^6 yr)
disappearance of deep convective envelope, rapid contraction (10^6 yr)
major phase of core helium burning (9×10^6 yr)
overall contraction with exhaustion of central helium
thick helium-burning shell
core neutrino loss, thin helium-burning shell

$M = 7 M_\odot$

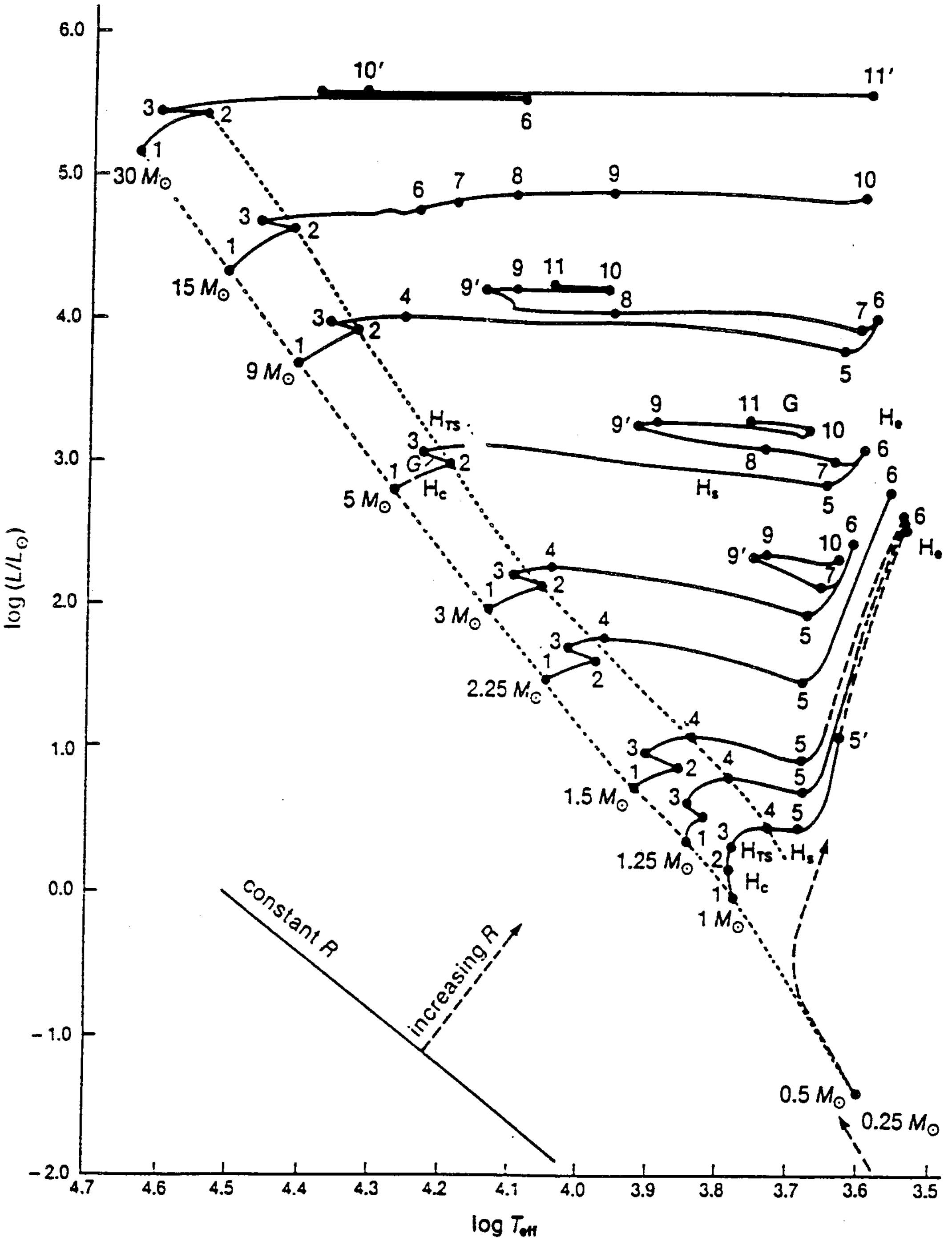


log L/L⊙
3.8
3.6
3.4
3.2
A
B
C
D
E
4.3
4.2
4.1
4.0
3.9
3.8
3.7
3.6
log Te
M_r/M
0.3
0.2
0.1
0
H→He
He
He→C
1.0
2.0
2.56
2.57
2.58
2.59
2.60
2.61
2.62
2.63
2.64
2.65
Alter i. 10⁷ Jahren



A
B
C
D
E
log r
13
12
11
10
9.1
1.0
0.5
0.4
0.3
0.2
0.1
0.05
0.02
0.01
0.005
0
1.0
2.0
2.56
2.57
2.58
2.59
2.60
2.61
2.62
2.63
2.64
2.65
2.66
Alter in 10⁷ Jahren

89c



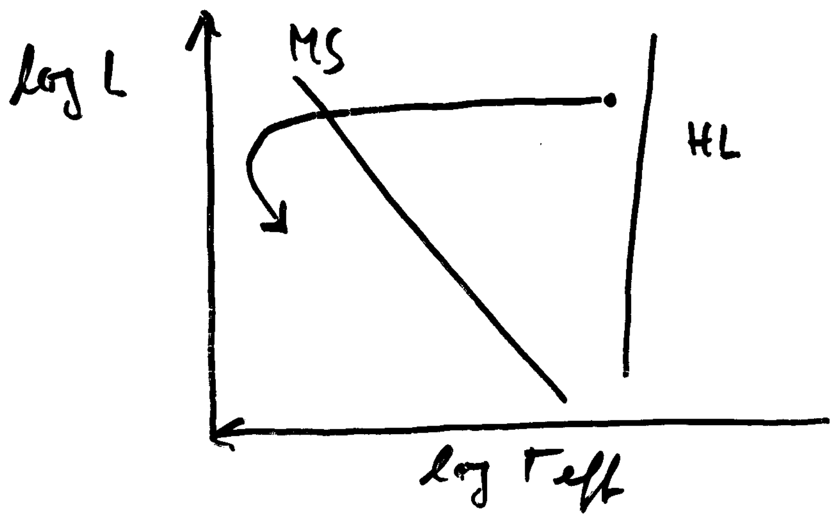
log (L/L☉)
log T_eff
30 M☉
15 M☉
9 M☉
5 M☉
3 M☉
2.25 M☉
1.5 M☉
1.25 M☉
1 M☉
0.5 M☉
0.25 M☉
constant R
increasing R
H_TS
H_c
H_s
H_e
G

Late stages of evolution

$M \lesssim 8 \dots 10\, M_\odot$

Degenerate cores
Red giants
Mass loss, Pulsation
Mira - variables
"Superwind"
Star loses envelope:
Planetary nebula

?



Degenerate core appears as
He-, C/O- white dwarf

$M \gtrsim 8 \dots 10\, M_\odot$

All burning phases are
reached, Si-burning
forms Fe-Ni-core

Stellar stability

Consider polytrope with $n=0$: ρ constant

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{GM_r}{r^2}$$

ρ constant : $$M_r = \rho \frac{4\pi}{3} r^3$$

$$\frac{\partial p}{\partial r} = -\frac{4\pi}{3}\rho^2 G r$$

boundary condition $p=0$ for $r=R$

$$p = \frac{2\pi}{3}G\rho^2 (R^2-r^2) = \frac{2\pi}{3}G\rho^2 R^2 (1-(r/R)^2)$$

Consider homologous variation of radius
(relative structure is not varied; r/R constant)

$$p \sim R^{-4} \quad \text{(each position)}$$

Pressure – Radius relation for equilibrium

Pressure provided by thermal pressure

Assumption : Homologous variation sufficiently fast $\Rightarrow$ changes of state adiabatic

$$p \sim \rho^{\gamma_{ad}} \sim R^{-3\gamma_{ad}}$$

Equilibrium only for $3\gamma_{ad} = 4$

$3\gamma_{ad} > 4$ Thermal pressure increases faster than equilibrium value $\Rightarrow$ restoring force, stability

$3\gamma_{ad} < 4$ Thermal pressure increases more slowly than equilibrium value, pressure deficit, contraction will be amplified, instability

Criterion for stability of mechanical equilibrium of a star (dynamical stability)

$$\gamma_{ad} > 4/3$$

$\gamma_{ad} < 4/3$ star unstable on dynamical timescales, enters dynamical phase

Exact analysis ($\gamma_{ad}(r)$)

$$\overline{\gamma_{ad}} := \frac{\int_0^R 4\pi r^2 \gamma_{ad} p \, dr}{\int_0^R 4\pi r^2 p \, dr} < 4/3$$

sufficient for instability

White dwarfs

Final stage of stellar evolution
Pressure provided by degenerate electrons

$$P \sim \mu_e^{-5/3} \rho^{5/3} \qquad \mu_e = \frac{n_{nuclei}}{n_e}$$

non relativistic electrons

$$P \sim \mu_e^{-4/3} \rho^{4/3}$$

relativistic electrons

Polytropes

$$R \sim M^{\frac{1-n}{3-n}}$$

$$\rho_c \sim M^{\frac{2n}{3-n}}$$

non relativistic electrons : $n = 3/2$
relativistic electrons : $n = 3$

Non-relativistic electrons:

$$R \sim M^{-1/3} \qquad \rho_c \sim M^2$$

Mass-Radius relation of white dwarfs

Increase $M \longrightarrow \rho_c \uparrow, \ \frac{N_e}{V} \uparrow, \ \frac{N_e}{V} \sim p_0^3$

$$p_0 \uparrow, \ \varepsilon_{max} \uparrow$$

Consequence : With increasing M transition to relativistic electrons

Consequence: Mass fixed

$$M = M_{ch} = \frac{5.76\, M_\odot}{\mu_e^2}$$

Apart from μ_e the Chandrasekhar-limit depends only on fundamental constants

$\mu_e = 2$ for ^4He, ^{12}C, ^{16}O, ^{20}Ne, ^{24}Mg!

$$M_{ch} = 1.44\, M_\odot$$

Physical meaning:

$M \neq M_{ch}$: No hydrostatic equilibrium

$M < M_{ch}$: Pressure dominates gravity
star expands $\rightarrow$ non relativistic deg.
$\rightarrow$ equilibrium

$M > M_{ch}$: gravitation dominates pressure
Star collapses

M_{ch} is the maximum mass for (electron) degenerate stars, i.e., for white dwarfs

Szenarios

1) – Binary system containing WD (He, C/O, ...)
– Mass transfer to WD
– Collapse when $M \approx M_{Ch}$ is reached
– Ignition of He-, C/O-burning under degenerate conditions, instability
– Thermal runaway, thermonuclear explosion
– Probably disruption details?
– Production of heavy elements (^{56}Fe ...)
– Supernova I, standard candles(?)

2) – Massive stars reaching Si-burning
– Due to lack of exothermic nuclear reactions Fe-Ni core contracts and becomes degenerate
– Si-burning increases Fe-Ni-core until $M \approx M_{Ch}$ is reached or
– M_{Ch} is reduced by inverse β decay

$$p + e^- \to n + \nu$$

correspondingly for nuclei $\Rightarrow \mu_e \uparrow$
– Collapse, no thermonuclear explosion
– Photo-dissociation of nuclei; e.g.

$$^{56}_{26}Fe \to 30n + 26p$$

inverse β decay

$$p + e^- \to n + \nu$$

- Photodissociation implies
  $\bar{\gamma}_{ad} < 4/3$
- Dynamical instability
- Collapse stopped by "neutron pressure"
- Formation of neutron star
- Timescale for collapse ~ sec
- Envelope collapses onto neutron star
- Shock wave, explosion, SN II
- energy balance

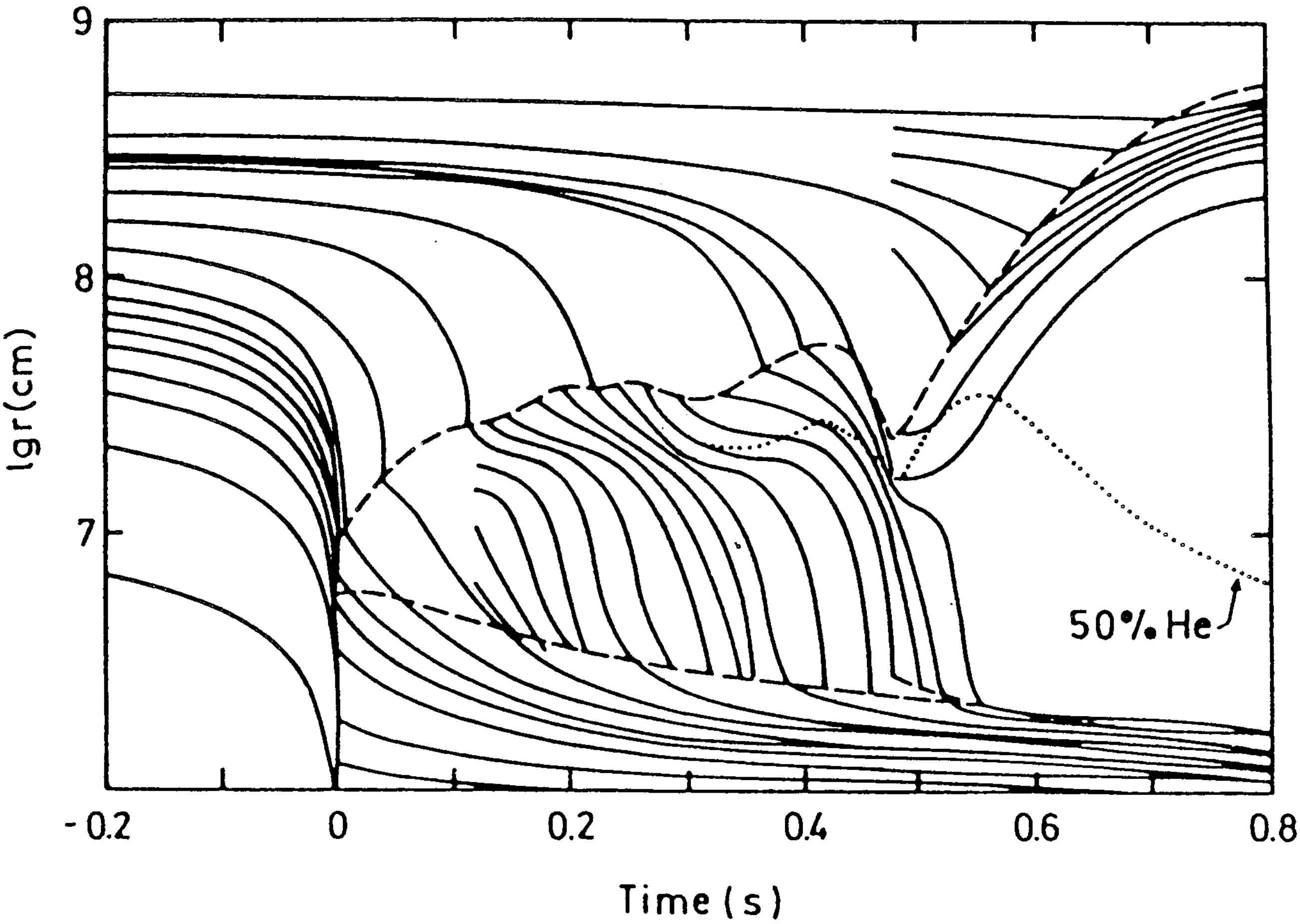
  gravitational energy released $\sim 10^{53}$ erg

  gravitational energy of envelope $\sim 10^{50}$ erg

  Energy of shock wave $\sim 10^{51}$ erg

  Radiation $\sim 10^{51}$ erg

  Neutrinos $\sim 10^{52 \ldots 53}$ erg



from: Kippenhahn & Weigert

Neutron Stars

Formation by Supernovae
e.g. Crab nebula

Pressure (first approximation): Degeneracy of neutrons

TOV-limit: analogous to Chandrasekhar limit

General relativistic corrections $\approx$ 20%
Radii: $\sim$ 10 km

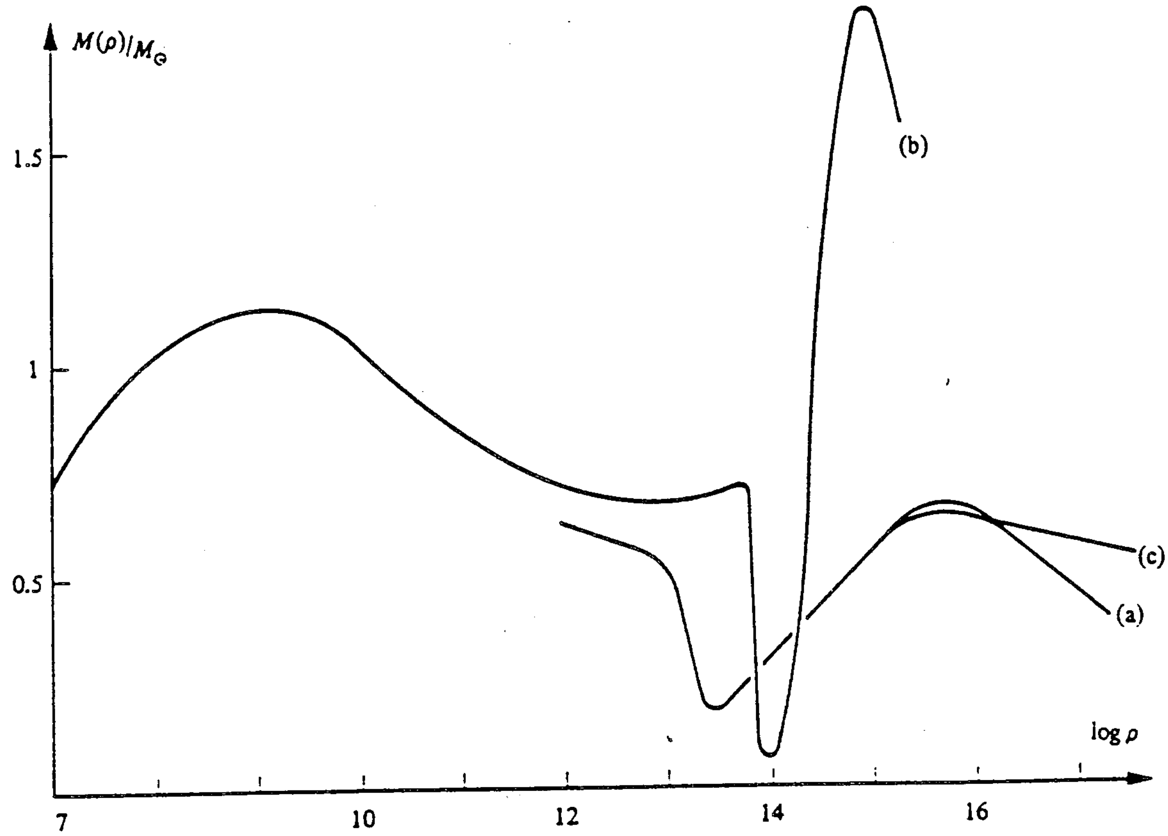
Problem: Equation of state: nuclear densities

$$\left.\begin{array}{lll} \text{Magnetic fields} & : & B \leq 10^{12}\,G \\ \text{Rotation} & : & P_{rot} \approx \text{msec} \end{array}\right\} \begin{array}{l} \text{flux/} \\ \text{angular momentum} \\ \text{conservation} \end{array}$$

M-ρ-diagram:

$$\frac{dM}{d\rho} > 0 \;\hat{=}\; \bar{\gamma}_{ad} > 4/3 \quad \text{stability}$$

$$\frac{dM}{d\rho} < 0 \;\hat{=}\; \bar{\gamma}_{ad} < 4/3 \quad \text{instability}$$



$M(\rho)/M_\odot$
1.5
1
0.5
(b)
(c)
(a)
$\log \rho$
7
10
12
14
16

Oscillations and Pulsations of stars

The star as an acoustic cavity with eigenfrequencies, modes etc analogous to organ pipe

$$\nu \cdot \lambda = c_{sound}$$

organ pipe : $\lambda = l/n$

star : $\lambda = R/n$

n: number of nodes, order of overtone

$$c_{sound}^2 = \gamma_{ad}\, P/\varrho$$

Eigenfrequencies of a star

$$\nu \sim \frac{n}{R} \sqrt{\gamma_{ad}\, P/\varrho}$$

Hydrostatic equilibrium

$$\frac{1}{\varrho} \frac{\partial p}{\partial r} = - \frac{G M_r}{r^2}$$

Estimate

$$P/\varrho \sim \frac{GM}{R}$$

$$\nu \sim n \left(\gamma_{ad}\, G\, \bar{\varrho}\right)^{1/2}$$

Fundamental mode: $n = 1$

Period Π:

$$\Pi \, \bar{\varrho}^{1/2} = \text{constant}$$

Period - density - relation

$$\Pi \, (\bar{\varrho}/\bar{\varrho}_\odot)^{1/2} = Q \quad ; \quad 0.^{d}03 \lesssim Q \lesssim 0.^{d}12$$

Excitation of Pulsations by Carnot-type processes

κ-mechanism: requires special temperature dependence of κ

compression phase: κ larger

damming up radiation $\rightarrow$

extra pressure $\rightarrow$ excitation

κ-mechanism restricted to narrow temperature range: $T_{eff}(L)$ Cepheid-strip

Stefan-Boltzmann: $L \sim R^2 T_{eff}^4$

Mass-luminosity-relation: e.g. $L \sim M^3$

$$\Rightarrow R(L), \; M(L)$$

with $\Pi \, \bar{\varrho}^{1/2} = \text{constant}$

Period-luminosity-relation

$$\Pi(L)$$

Cepheids: $\Pi \sim O(d)$: important distance indicators

Nucleosynthesis

- H, He : primordial
- iron group elements and below: thermonuclear fusion in stars
- Synthesis of elements heavier than "iron":
  Neutron capture (binding energy per nucleon, Coulomb barrier)
- Abundances : double maxima :
  2 processes

- r - process rapid neutron capture SN II
  s - process slow " " Giants

- competition between β decay $\leftrightarrow$ n-capture

- number of nuclei exposed to neutron source

$$N \sim 1/\sigma \qquad \sigma: \text{cross section for capture}$$

- s-process : capture slow compared to decay
  proceeds in "stability valley"
- r-process : proceeds outside "stability valley"
  stop of exposure $\rightarrow$ β decay
  distribution shifted towards
  smaller number of neutrons

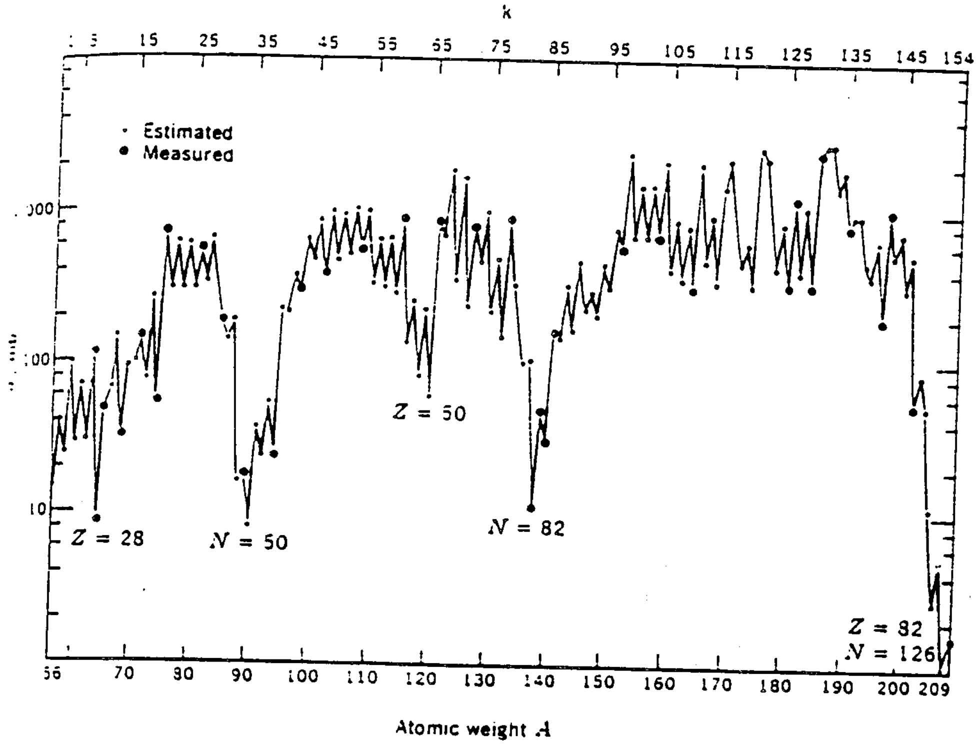


Fig. 6.2. Neutron-capture cross-sections at energies near 25 keV. Very large dips occur at the magic numbers. After D.D. Clayton, *Principles of Stellar Evolution and Nucleosynthesis*. McGraw-Hill. 1968 and University of Chicago Press 1984. p. 556. ©1984 by the University of Chicago. Courtesy Don Clayton.

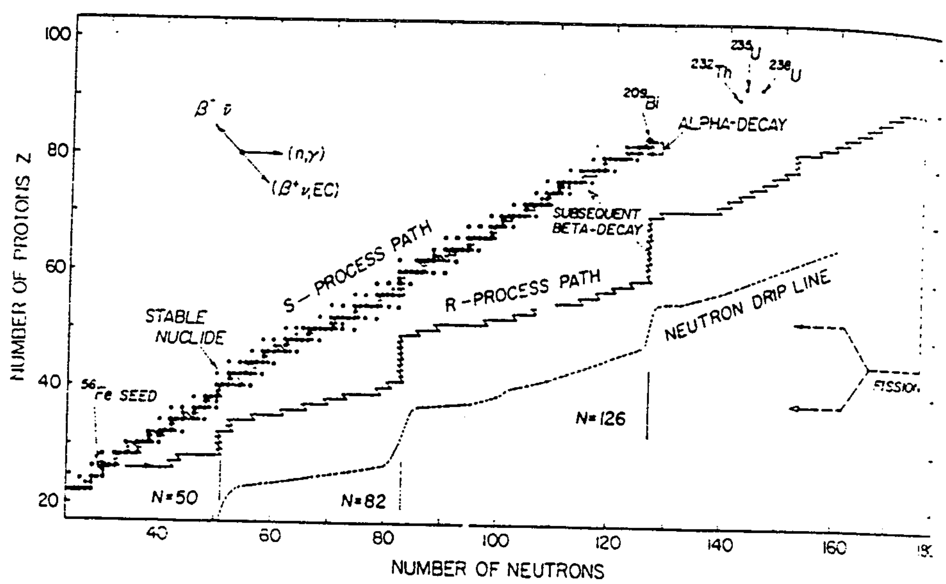


Fig. 6.9. Neutron capture paths in the N, Z plane. The r-process path was calculated for a temperature of 10^9 K and a neutron density of 10^{24} cm^{-3} (Seeger, Fowler & Clayton 1965). The dotted curve shows a possible location of the neutron drip line after Uno, Tachibana & Yamada (1992). Adapted from Rolfs & Rodney (1988).

Point sources of radiation (stars)

observed Flux $\frac{L}{4\pi D^2} = F$

L : luminosity D : distance

Apparent brightness (magnitude):

$$m \sim \log F$$
$$m = -2.5 \log F + \text{constant}$$

Dependence on wavelength

$$m_\lambda, \; m_{pg}, \; m_V$$

Filters: U B V R I

For total energy radiated: bolometric magnitude

$$m_{bol}$$
$$m_{bol} = m_{vis} - BC$$

Colour, colour index:

$$CI = m_{\lambda_1} - m_{\lambda_2}$$

E.g.: $B-V$

Correction for distance:
Absolute brightness (magnitude)
=: apparent magnitude of object
in 10 pc distance

$$M = -2.5 \log F^{10\,pc}$$

$$m - M = -2.5 \log \left(\frac{10\,pc}{D}\right)^2 + \text{const.}$$

Distance modulus:

$$m - M = 5 \log D[pc] - 5$$

Absolute bolometric brightness:

$$M_{bol} = -2.5 \log L/L_\odot + 4.72$$

Colour: indication for effective temperature T_{eff}

Consider black body radiation as zeroth approximation

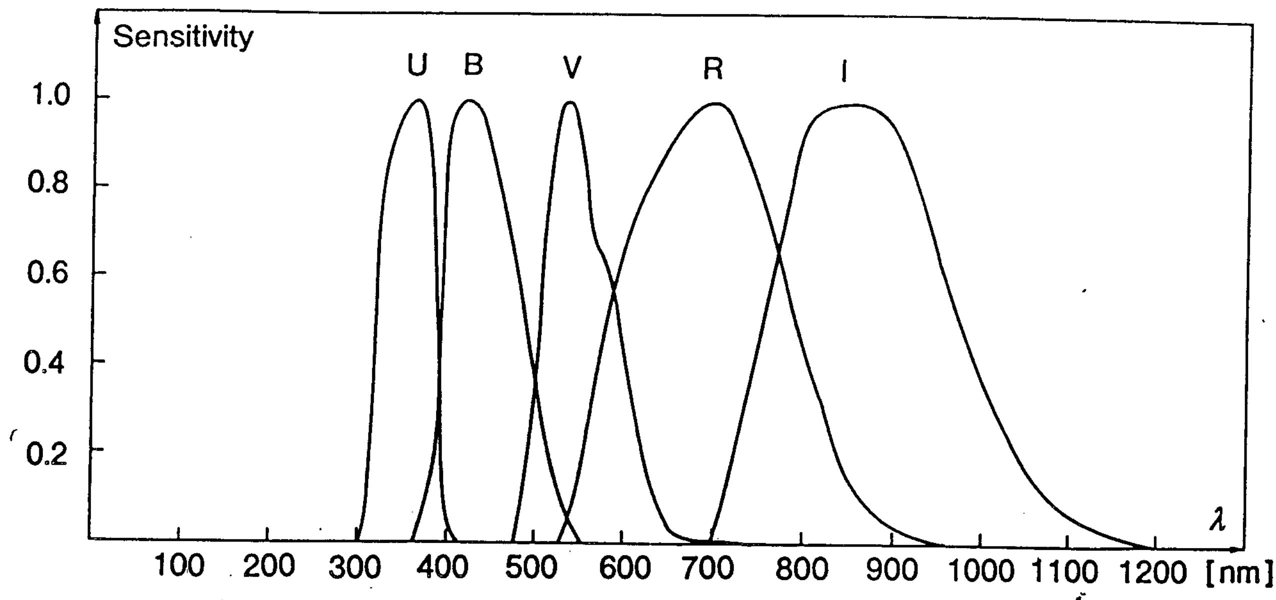
Relation spectral type – Colour – T_{eff}.

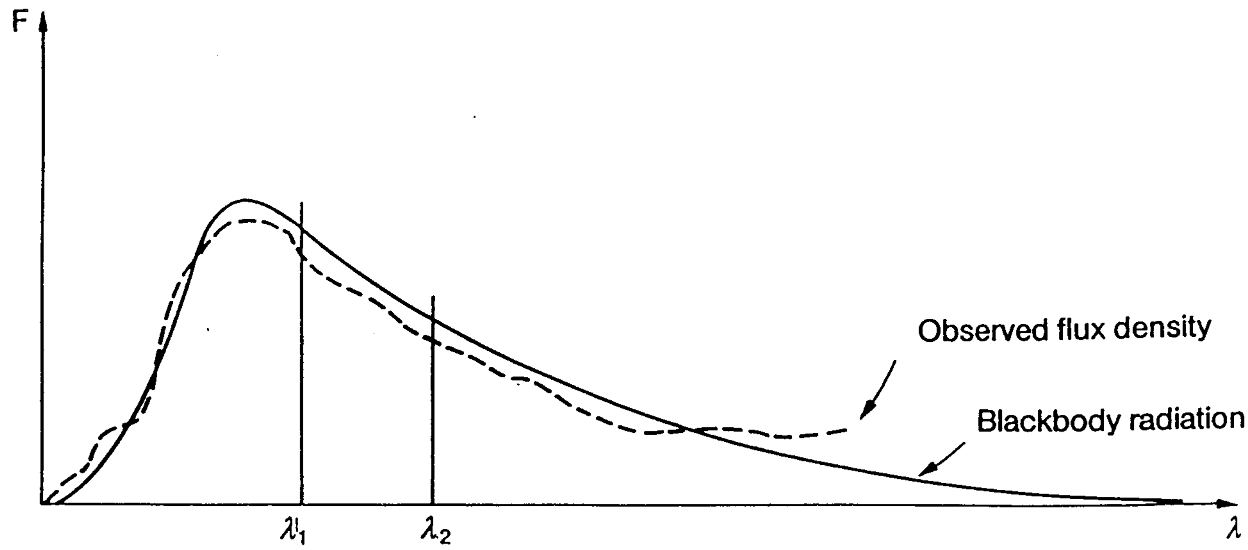
Example: Harvard-classification

Stellar Parameters

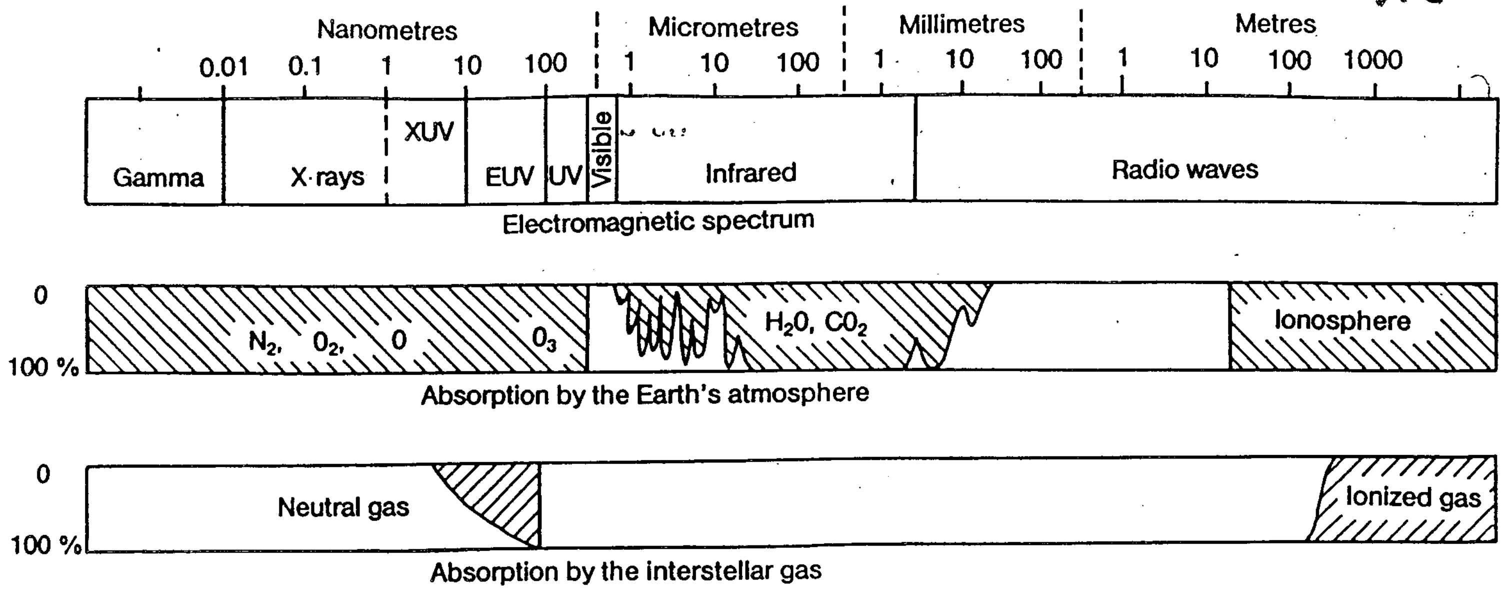
- Luminosity
- (Effective) Temperature
- Radius
  - Interferometry
  - Lunar occultations
  - Eclipsing binaries
  - with L, T_{eff} from $L = \sigma 4\pi R^2 T_{eff}^4$
- Mass
  - Binaries
  - spectroscopic, eclipsing
  - Kepler's 3rd law
  - Spectroscopy: $M \sin^3 i$
- Chemical composition: Spectrum
- Rotation, magnetic fields: Spectrum

Aa

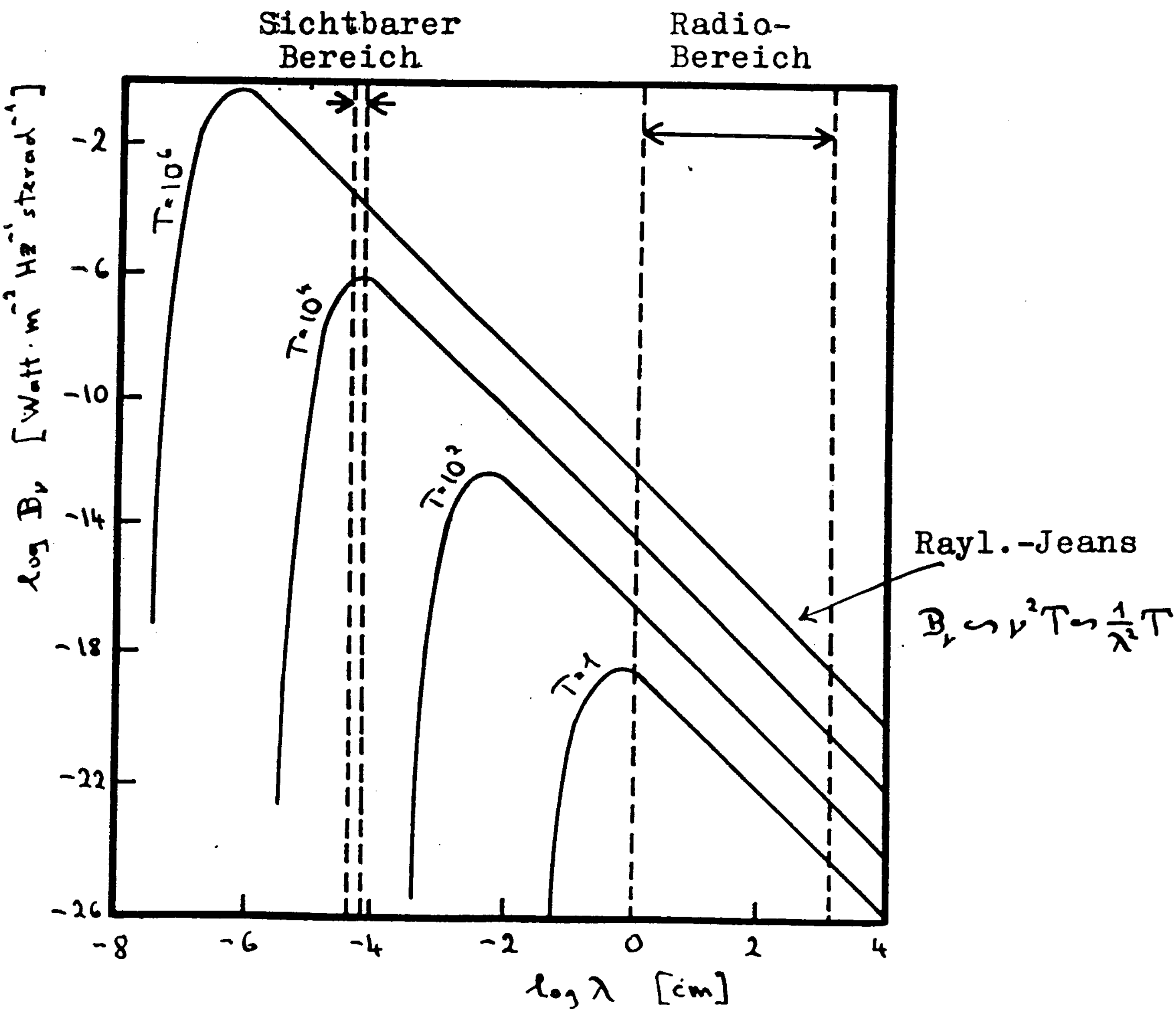

Sensitivity
U
B
V
R
I
1.0
0.8
0.6
0.4
0.2
λ
100
200
300
400
500
600
700
800
900
1000
1100
1200
[nm]


F
Observed flux density
Blackbody radiation
λ1
λ2
λ

Ab



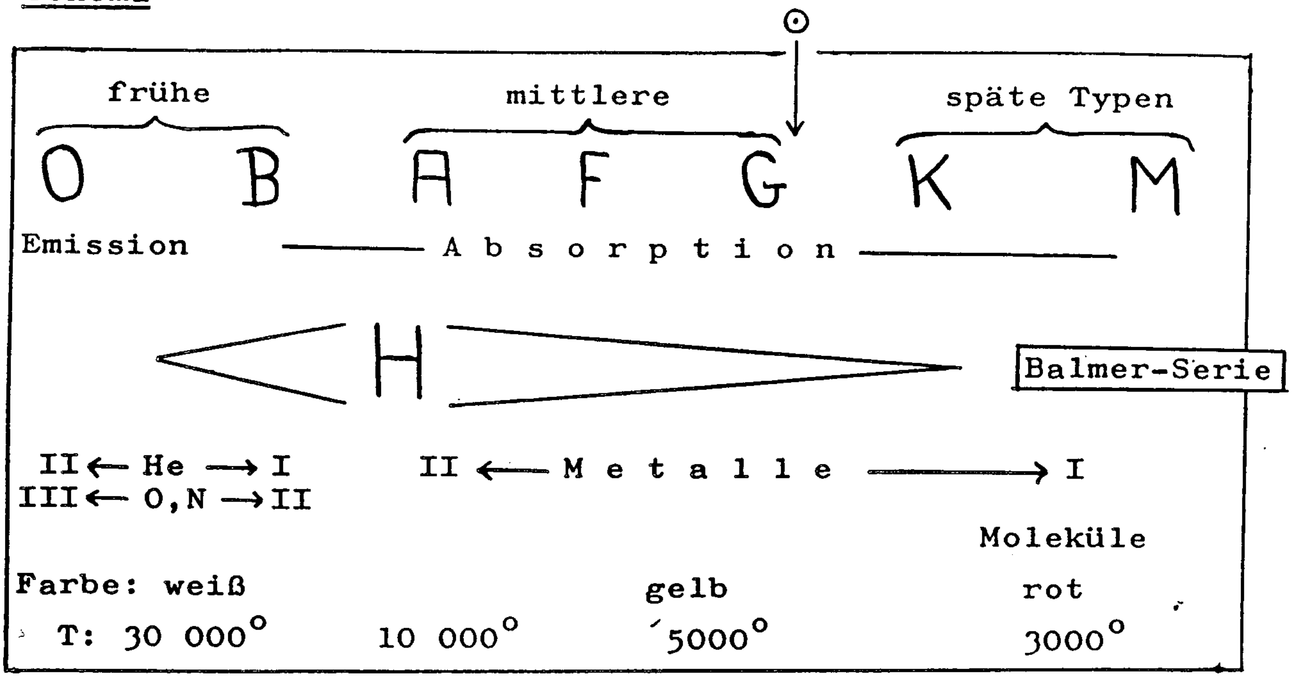
Nanometres
Micrometres
Millimetres
Metres
0.01
0.1
1
10
100
1
10
100
1
10
100
1
10
100
1000
Gamma
X-rays
XUV
EUV
UV
Visible
Infrared
Radio waves
Electromagnetic spectrum
0
100 %
N_2, O_2, O
O_3
H_2O, CO_2
Ionosphere
Absorption by the Earth's atmosphere
0
100 %
Neutral gas
Ionized gas
Absorption by the interstellar gas



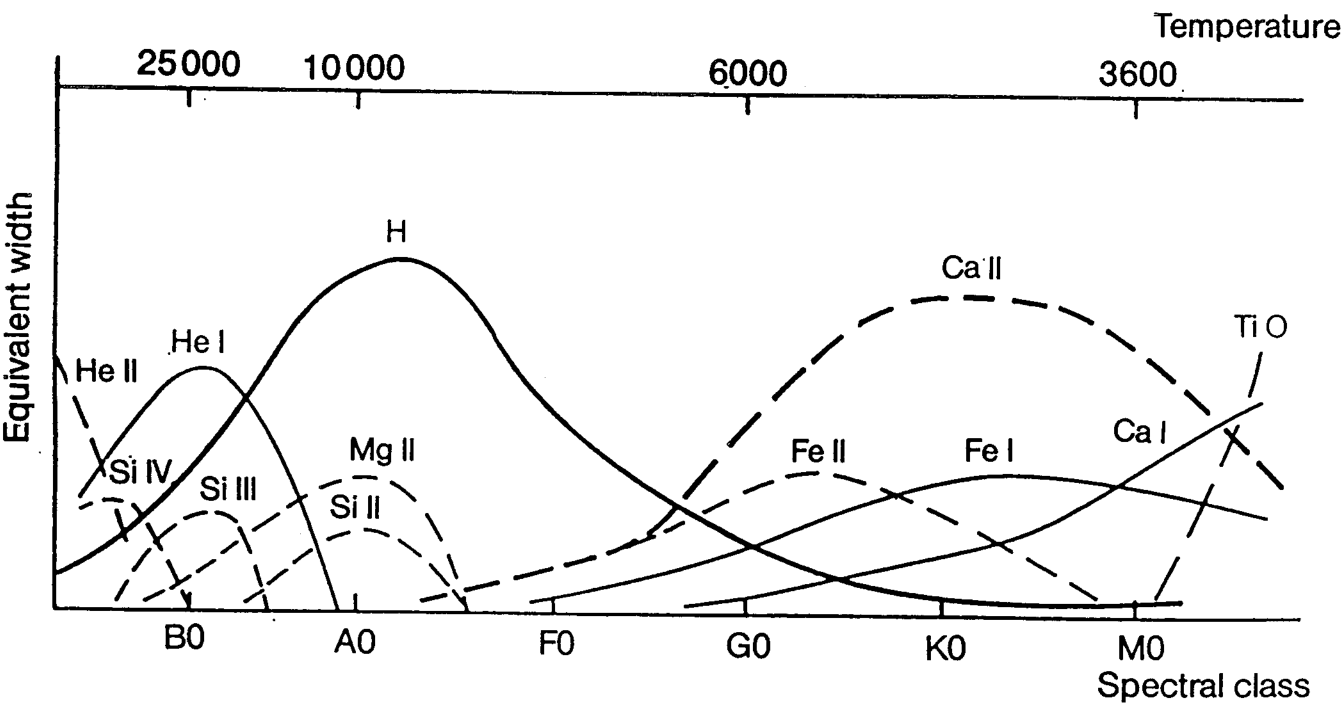
Sichtbarer Bereich
Radio-Bereich
log B_ν [Watt · m^{-2} Hz^{-1} sterad^{-1}]
-2
-6
-10
-14
-18
-22
-26
T=10^6
T=10^4
T=10^2
T=1
-8
-6
-4
-2
0
2
4
log λ [cm]
Rayl.-Jeans
$B_\nu \sim \nu^2 T \sim \frac{1}{\lambda^2} T$

Ac

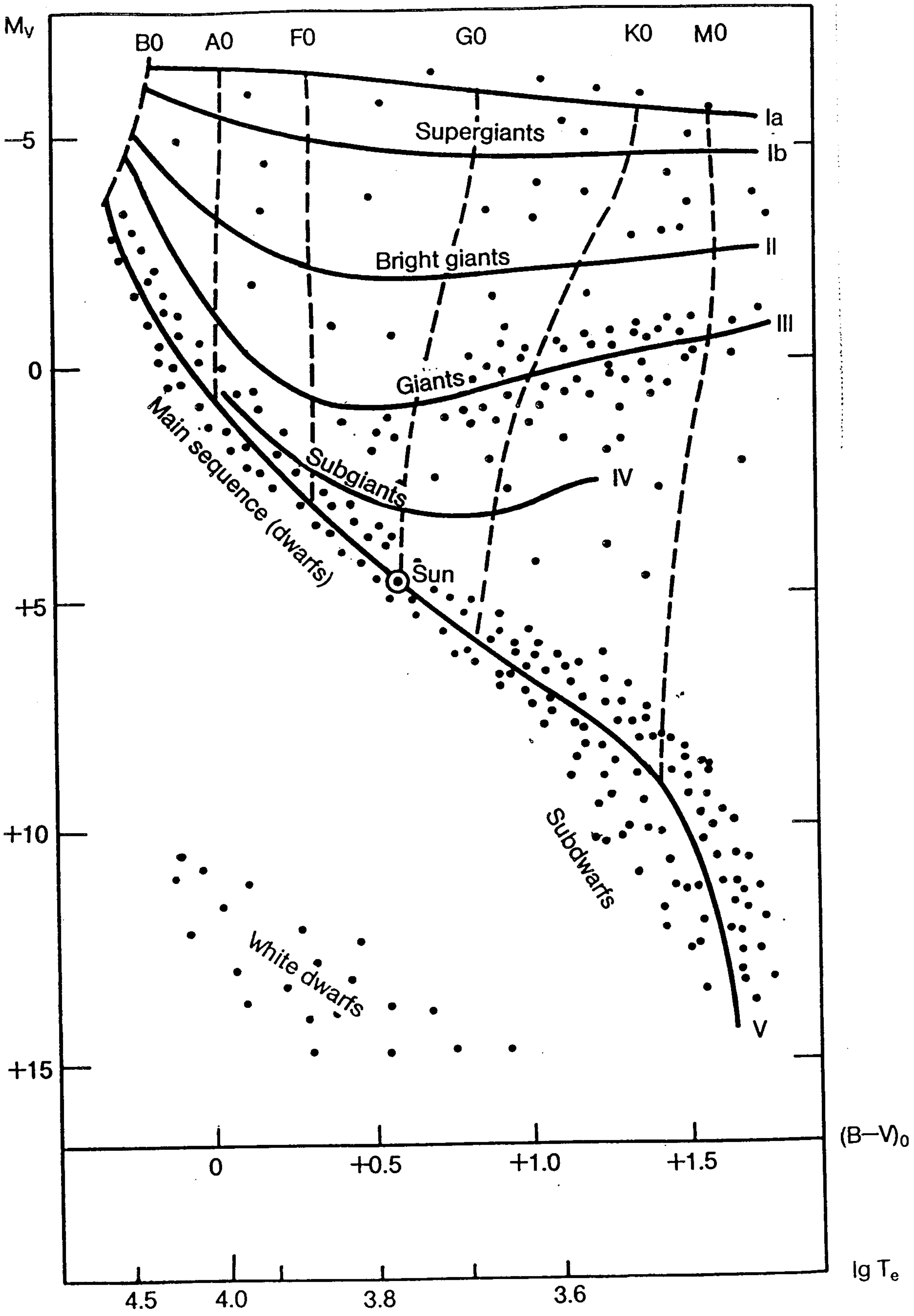
Schema



I: neutral; II: einfach ionisiert; III: zweifach ionisiert



Ad



M_V
B0
A0
F0
G0
K0
M0
Ia
Ib
II
III
IV
V
Supergiants
Bright giants
Giants
Subgiants
Main sequence (dwarfs)
Sun
Subdwarfs
White dwarfs
−5
0
+5
+10
+15
0
+0.5
+1.0
+1.5
$(B-V)_0$
4.5
4.0
3.8
3.6
lg T_e